Seminar zur Algebraischen Geometry program :
A path towards semistable models of smooth projective curves over $\mathbb{Q}$

Important informations :

This seminar will be hold during the winter semester 2026/27, on Tuesdays, between 2 :00 - 4 :00 p.m. in SR 1C. Careful that my French habits will make the seminar start at 2 :00 p.m. sharp, and end somewhere between 3 :30 and 4 :00 p.m. depending on the week.

You should register online to the lesson if you want to validate it. Here is the link : `https://www.uni-muenster.de/LearnWeb/learnweb2/course/view.php?id=95213`. We can have a meeting so that I can give details about what I expect from the talk, help you naviguate the references or solve some issues. Ideally, this meeting would take place on Tuesday afternoon after the previous talk, or the following wednesday morning.

Do not hesitate to contact me by email for scheduling the appointment or for questions. My email is n.marquis@uni-muenster.de and you can also find all informations on my webpage `https://nmarquis-maths.github.io/teaching.html`.

Finally, if you don't find the book, or .pdf of some reference, please ask me. You can also discuss with me if you think you need to adapt the length to be able to explain properly rather than hastily, so that we can maybe make a selection together of what's the most important. The things labelled "if time allows" should only be covered if the program reveals itself too short, and I expect this will not happen.

Talk 1. (Oct. 13, Lilly Steinbauer) Journey to the definition of Dedekind schemes

Aim : we will walk on an avenue of definitions and recalls to reach the definitions of Dedekind schemes of dimension 1 *as well as regular schemes.*

The aim of this first talk is to recall and/or define some very useful properties of rings and schemes that we will be using in the whole seminar. As some may have already seen some of the notions, try to focus on giving formal definitions and intuition via examples rather than running all proofs. I only require the proof of [Sta23, Tag 0357] or of [Sta23, Tag 01ON] using [Sta23, Tag 01OM] to be done (but feel free to add other if you have time and think it is useful). I give mostly *Stacks Project* as reference : it is obviously too thick, hence try to extract what seems the essential for each notion.

We define the notions of integral ([Sta23, Tag 01OJ]) and normal schemes ([Sta23, Tag 037B] for ring theoretic definitions, [Sta23, Tag 033H]) and a few of their characterisation. We then define the dimension of a scheme ([Sta23, Tag 00KD]). You can state just enough properties to then run the examples of $\mathbb{A}^n_k$ say for an algebraically closed field k and for $n \geq 2$, of the "cross" $\mathrm{Spec}(k[x,y]/xy)$ and of $\mathrm{Spec}(\mathbb{Z})$. Define Dedekind schemes (nothing more than [LE02, §4.1, Def 1.2]) and give examples. End by the definition of regular schemes (select among [Sta23, Tag 065U] [Sta23, Tag 02IR]).

Only if time allows, try to understand why [Sta23, Tag 0B2I] roughly says that a two dimensional irreducible scheme over a Dedekind scheme of dimension 1 can be thought as a family of curves. This can be inserted before defining regular schemes.

Talk 2. (Oct. 20, Sevag Buyuksimkesyan) Projective geometry and smoothness

Aim : we will define and give intuition about projective and smooth schemes/maps.

As projective schemes have been covered in previous classes (though I think it is useful to clear the minds about it) and smoothness is a wide subject, I expect this talk to give precise definitions and intuition rather than proofs if one has to choose. I only require the proof of [LE02, §4.1.1, Lem. 1.18] to be done (but feel free to add other if you have time/ think it is useful). The reference I give are obviously too thick, hence try to extract the essence and useful properties for each notion. You may use the full two hours.

We first define projective scheme associated to a graded ring ([Duca, §6.1] is the clearest reference in my mind, but as it is in French, feel free to find your own). Make a quick analysis of $\mathbb{P}^N_k$, its closed subschemes and of the invertible sheaves $\mathcal{O}_{\mathbb{P}^N_k}(n)$ (see [Duca, §6.2.1-6.2.4, §6.3.10] or [Sta23, Tag 01ND] [Sta23, Tag 01MM]). We then define projective morphisms ([Sta23, Tag 01W7]), which allows to define an arithmetic surface (only [LE02, §8.3.1, Def. 3.1 and 3.14]). Please mention the criterion [LE02, §4.1.1, Lem. 1.18] for being normal.

The last task is to define the notion of a smooth map. I think it is disturbing as a first approach to give the definition using the naive cotangent complex. Rather combine [Sta23, Tag 00T6] [Sta23, Tag 01V7] [Sta23, Tag 01V8] to give a definition that looks more like a jacobian criterion. This will fit better the intuition that a (smooth) differential manifold is locally an affine space, but that algebraically we do not have the inverse function theorem, hence we "merely" put the jacobian criterion. Give useful examples and mention [Sta23, Tag 00TN]. Give a criterion for finite type scheme over a field (i.e. for the fibers) following [LE02, §6.2.1, Prop. 2.2]. Also mention [LE02, §6.2.1, Prop. 2.5] for further use.

Talk 3. (Oct. 27, Adéla Heroudková) Picard group and divisors topics

Aim : we will recall/define (depending on what people have already seen) Picard group, Cartier divisors, effective Cartier divisors, Weil divisors and the link between them.

As it is a lot to cover, I do not expect all the proofs. Try to give at least a few, but mainly on stating properly the definitions and main results, as well as giving an intuition of how the notions behaves.

We first define the Picard group of a scheme. For that you can look at [LE02, §5.1, Exer. 1.12], [Duca, §3.3.1-3.3.6] which is very well written but in French, or take hints from [Sta23, Tag 01CR] being careful that it is written in the broader setup of ringed spaces. We then define effective Cartier divisors [Sta23, Tag 01WR] [Sta23, Tag 01WS] [Sta23, Tag 01WT]. Give (without proof) the definition of the sheaf of stalks of meromorphic functors $\mathcal{K}_X$ in [LE02, §7.1.1], the definition of Cartier divisors [LE02, §7.1.2, Def. 1.17] and explain how the two definitions of effective Cartier divisors coincide. Reach the definition of Cartier class group and its comparison with Picard group [LE02, §7.1.2, Cor. 1.19] in the reduced case. Follow [LE02, §7.2.1] to define Weil divisors and the class group, then compare everything to the Cartier version [LE02, §7.2.1, Prop. 2.14 and 2.16]. You can skip all the projective examples.

Talk 4. (Nov. 3, Jakob Gerth) What does it mean to blow up... and some genus too

Aim : we will define and study the notions of arithmetic and geometric genus then of blow-ups.

First define arithmetic and geometric genus of a projective curve over a field. For that, investigate from [LE02, §7.3.2, Def. 3.19] and [Sta23, Tag 0BY6]. Note that each reference talk about dualising sheaf/canonical sheaf $\omega_{Y|k}$ which definition you can find in [LE02, §6.4.2] but for a smooth projective curve over a field this is merely the sheaf of differentials $\Omega^1_{Y|k}$, locally free of rank 1. Then, try to run some examples of genus computations.

In a second time, we want to define the blow-up of a closed subscheme and its universal property. Follow [Sta23, Tag 01WS]. I especially want you to state (and maybe prove) [Sta23, Tag 02OS] [Sta23, Tag 0806] and the first part of [Sta23, Tag 02NS].

Note : you need to do a 1h30 sharp talk as the last half hour will be devoted to running out together computations of blow-ups.

Talk 5. (Nov. 10, Nataniel Marquis) Birational maps and their description by blow ups

Aim : we will define ample invertible sheaves that make a very useful link between divisors and projective maps. We will then use this notions to study birational morphisms, essentially morphisms of schemes over our Dedekind scheme which are model of the same scheme. We will prove that these maps are blow-up maps.

We first study ample and very ample invertible sheaves. We can in particular state the relation of projectivity with these notions and the second part of [Sta23, Tag 02NS]. For use in the proof of [LE02, §8.3.3, Prop. 3.30] (and other) we need [LE02, §7.5, Prop. 5.5] and the ability to make an ample sheaf very ample by multiplication.

We then define birational maps and morphisms ([Sta23, Tag 01RN] [Sta23, Tag 01RR]. To help yourself, you can remember that we will can about fibered surface, which are integral and hence can be assumed irreducible. The second part of the talk is to follow [LE02, §8.1.3] to reach [LE02, Th. 1.24]. We will fill what was leggitly cut from previous talk to reach the aforementionned theorem.

Talk 6. (Nov. 17, speaker not fixed) Models can be normies

Aim : we finally define properly what we mean by a model of a normal projective curve over the fraction field of a Dedekind scheme of dimension 1*, then explain the existence of normal/regular models for when the curve is smooth and the Dedekind scheme affine.*

For that, one first gives [LE02, §10.1.1 Def. 10.1], then move directly to proving the existence of normal models for $S = \mathrm{Spec}(A)$ affine by examining [LE02, §10.1.1 Ex. 10.4]. One may want to understand (and simplify in the affine case) the link between projectivity and the description as closed subscheme of $\mathbb{P}^1_{\mathrm{Frac}(A)}$ given in [Sta23, Tag 087S]. The example provides us a model, then we explain how to obtain a normal model by normalising. Try to use [Sta23, Tag 0BG9] even if you don't prove it. To move on to regular models, on need the theory of desingularisation. This is a construction that is not that hard to describe, even though the proof of its correctedness is too hard for this seminar. In the talk, we will hence only state the definition [LE02, §8.3.4 Def. 3.39], then explain the construction before [LE02, §8.3.4 Th. 3.44] and state it. Maybe try to avoid talking about excellent scheme and rather explain why the singular locus is closed for a normal model. You can also get inspiration and maybe clearer (but more restrictive) statements in [Sta23, Tag 0C2R]

The second part of the talk will try to find an interesting example and run some explicit computations.

Talk 7. (Nov. date to fix, Haixu Wang) Intersection theory on surfaces

Aim : we study intersection theory on surfaces, giving a combinatorial date about how two divisors in a surface intersect. This seminar will further use it to derive combinatorial date from the special fibers of models.

We first define various notions of multiplicities and degrees and try to link them together : see the definition of multiplicity of a Cartier divisor at a closed point of codimension 1 in [LE02, §7.1.2 Eq. (1.2)], the degree of a Cartier divisor [LE02, §7.3.1 from 3.1 to 3.5]. We finish by the degree of line bundles [LE02, §7.3.2 Def. 3.39, Lem. 3.30] (you can take Riemann theorem as a black box if you don't have time). Run the example of $\mathbb{P}^1_k$.

The second part of the talk consists on introducing intersection theory on surface to pave the road afterwards. We advice to pick what seems reasonable among [LE02, §9.1-9.2], focusing on stating [LE02, §9.1 Th. 1.12] and [LE02, §9.2 Prop. 2.5].

Talk 8. (Dec. 1, Gun Kongarttagarn) These divisors are really exceptionnal or how to reach minimal models

Aim : explain how to obtain minimal regular models (roughly every regular models have a birational morphism to them) by contracting a specific type of curve.

Define the notion of contraction of a family of integral vertical curves [LE02, §8.3.3 Def. 3.27]. We aim to reach the criterion [LE02, §8.3.3 Prop. 3.30 (ii)⇒ (i)] and proving it quite carefully, which need to go up accordingly in the loc. cit. book.

We then brush [LE02, §9.3.1] defining the notion of exceptionnal divisors, i.e. divisors that we can contract (hence proving that the surface is not minimal). The paragraph explain that it is always the blow-up at a point, using results of Talk 5 and compute everything in this case, showing that $E \simeq \mathbb{P}^1_{k'}$ and $E^2 < 0$. It shows that we can always contract such divisor but that regularity is not necessarily preserved, ending up with a characterisation of such exceptionnal divisors called Castelnuovo's criterion [LE02, §9.3.1 Th. 3.8].

We then investigate the notion of relatively minimal surfaces by brushing [LE02, §9.3.2], analysing how to obtain representatives of birational morphisms classes [LE02, §9.3.2 Prop. 3.19] by successively contracting exceptionnal divisors.

Finally, we brush [LE02, §9.3.3] which covers the notion of minimal model. It boils down to proving that two relatively minimal models are isomorphic [LE02, §9.3.3 Th. 3.21].

As the three last paragraphs represent a lot to cover, we do not expect to make all the proof. Try to quote all the results mentionned, and to prove in details one of the paragraphs at least.

Talk 9. (Dec. 8, speaker not fixed) Semistable models under scrutiny

Aim : we define semistable algebraic varities over $\mathbb{F}_p$, which are roughly the one étale locally looking like crosses $\mathrm{Spec}(k[x,y]/xy)$. We also study semistable models over $\mathbb{Z}_p$, which are roughly étale locally looking like annuli of integer length.

The first part of the talk is to define nodes, also called singular ordinary double points of a variety over an algebraically closed field. For this, we can mix [LE02, §7.5] (from the paragraph before Lem. 5.12 to Prop. 5.15) and [Sta23, Tag OC1P] and focus on double points. We will make appear the equivalence between the description by δ and branches invariants and the description of the stalks.

Then, we define semistable curves over a field, also called at-worst-nodal. One can compare [LE02, §10.3.1 Def. 3.2] and [Sta23, Tag 0C47]. Run out [LE02, §10.3.1 Ex. 3.4]. We can mention [LE02, §10.3.1 Prop. 3.7] if we have time. Finally, we reach the definition of a smooth projective curve having a semistable model, equivlently having at-worst-nodal families of curves as model. One can once again compare the compact are clear [Sta23, Tag 0CDB] [Sta23, Tag 0C58] and [LE02, §10.3.2] which uses less frightening terms. There is no need to cover everything in this paragraph, but it is important to highlight the étale local structure of at-worst-nodal families of curves [LE02, §10.3.2 Cor. 3.22] and the link with minimal models [LE02, §10.3.2 Th. 3.34] which both have analogues in Stacks Project.

Talk 10. (Dec. 15, William Hendareo) Proof of weak Deligne-Mumford part I : combinatorial invariants

Aim : we associate combinatorial data to the special fiber of an arithmetic surface, define two combinatorial invariants β and a "new" genus, and make the link with the arithmetic genus and the picard group.

Begin by stating a weak Deligne-Mumford theorem that will be the aim of the next two talks. It is [LE02, §10.4 Th. 4.3] for S being the spectrum of a discrete valuation ring and semistable instead of stable.

First, we dig into [LE02, §10.1.4] aiming to understand [LE02, §10.1.4 Ex. 4.56]. We advice to present things in a reverse order. First explain what data are given in [LE02, §10.1.4 Ex. 4.56] and their properties that will suit the definition of a type [LE02, §10.1.4 Def. 4.55]. For that, you can help yourself with [Sta23, Tag 0C5Y] (while being careful as the definition of type in Stacks is more involved). Then, one can define the type of (the special fiber) of an arithmetic surface [LE02, §10.1.4 Def. 4.55] and its genus. We finish this first part by constructing the graph associated with a type as well as the invariant β. This can be made by following or selecting among [LE02, §10.1.4] up to [LE02, §10.1.4 Prop. 4.51]. The first two items of this property should be explained.

In a second time, we investigate the link between the Picard group of an arithmetic surface X over a discrete valuation ring, its special fiber and invariants that has to do with the type. This is [LE02, §10.4.2] until [LE02, §10.4.2 Prop. 4.17].

Talk 11. (Dec. 22, Ernesto Thimm) Proof of weak Deligne-Mumford part II ;

Aim : we finish the proof of the weak Deligne-Mumford theorem by studying abelian and toric rank of a regular model.

We define the abelian and toric rank of an arithmetic surface by covering [LE02, §10.4.2 5.17-5.24]. Remember to restrict to arithmetic surface to not state/prove everything with the greatest generality and details.

Taking [LE02, §10.4.2 Th. 4.19] as a blackbox, we explain how to relate the toric rank and the construction of last talk in [LE02, §10.4.2 Prop. 4.20].

Finish the proof as in [LE02, §10.4.2 p.542].

If time allows, we can prove the genus 0 case using [Sta23, Tag 0CDK]. If time still allows, we can brush the genus 1 case extracting the core ideas from [Sta23, Tag 0CEG].

Talk 12. (Jan. 12, Gautham Mahesh) Formal schemes

From here, the seminar takes a turn : the upcoming talk are more cultural than formal (only the schemes are). It is useful to make precise definitions when possible, but it is also allowed to drop some technical adjectives in the presentation, possibly mentionning that your dropped them. I rather expect you to focus on why the objects were introduced, to compare them to the notions you know, and to give an intuition about how to think about them.

This talk first aims to give an idea about how formal geometry serves as the correct replacement for model over $\mathrm{Spec}(\mathbb{Z}_p)$ and how it can give rise to rigid geometry over $\mathbb{Q}_p$. Trying to focus on the case of p-adic formal schemes, we can freely get inspiration from [FK18, Intro §0, 3-6] and [Ill03, §1]. One should definitely highlight how we keep only the topological information of the special fiber while still keeping $\mathbb{Z}_p$-algebras (usually non torsion) as structure sheaves. Then, we will try to give a broad overview of [FK18, p. I 1.1, 1.2, 1.4, II 1.1], and give some (but not too much) clean definition. Finish by an analysis of the canonical formal scheme sent by localisation to the unit ball, i.e. the p-adic formal scheme

$$\mathrm{Spf}(\mathbb{Z}_p\langle t\rangle),\ \text{ where }\ \mathbb{Z}_p\langle t\rangle := \left\{ \sum_{n\geq 0} c_n t^n \,\middle|\, c\in \mathbb{Z}_p^{\mathbb{N}},\ c_n \xrightarrow[n\to+\infty]{0} \right\}.$$

Talk 13. (Jan. 19, Angwei Li) Rigid analytic spaces from Berkovich's point of view

From here, the seminar takes a turn : the upcoming talk are more cultural than formal. It is useful to make precise definitions when possible, but it is also allowed to drop some technical adjectives in the presentation, possibly mentionning that your dropped them. I rather expect you to focus on why the objects were introduced, to compare them to the notions you know, and to give an intuition about how to think about them.

This talk first aims to give the main features of Berkovich theory of rigid analytic spaces over $\mathbb{Q}_p$ and how it relates to the previous approach. To motivate, we can freely get inspiration from [Duc06, Intro and §1] and [FK18, Intro §7]. We advice to run a precise definition of a $\mathbb{Q}_p$-analytic Berkovich space but to keep other definition vague. In terms of reading and presenting, I advice to follow a precise path. First, read until [Duc06, Def. 1.4] to reach the definition of the topological space of an affinoid Berkovich space. We should then run out an explicit description of $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{Q}_p}$ as in [Poi16]. We can compare the classification of the points to the (more technical) one given in [Ducb, §3.3], if time allows. Give other examples without the full description like the polydisks [Duc06, Ex. 1.6], circles [Duc06, Ex. 1.3], the annuli [Duc06, Ex. 1.14]. We can then brush the actual construction of $\mathbb{Q}_p$-analytic Berkovich spaces and their structure sheaf by following [Duc06, §1.3 until Def. 1.24 excluded]. Finally, try to explain the generic (analytic) fiber of a finite type p-adic formal scheme as in [Duc06, §1.4].

Note that you can also pick among Berkovich original articles but they are less beginner friendly.

Note : the talk should be 1h 30 sharp as the last hour will be devotes to some computations together.

Talk 14. (Jan. 26, Nataniel Marquis) A pencil, drawing triangulations and you got your semistable model

You will discover the details in due time.

Références

[Duca] Antoine Ducros. *Introduction à la théorie des schémas*. Lecture notes available at webusers.imj-prg.fr/ antoine.ducros.

[Ducb] Antoine Ducros. *La structure des courbes analytiques*. draft of a book in preparation. URL : http://webusers.imj-prg.fr/~antoine.ducros/trirss.pdf.

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[FK18] Kazuhiro Fujiwara et Fumiharu Kato. *Foundations of Rigid Geometry I*. European Math. Soc., 2018.

[Ill03] Luc Illusie. *Grothendieck's existence theorem in formal geometry*. online notes for Advanced School in Basic Algebraci Geometry. 2003. URL : https://indico.ictp.it/event/a0255/session/14/contribution/9/material/0.

[LE02] Qing Liu et Reinie Erné. *Algebraic Geometry and Arithmetic Curves*. Oxford University Press, mai 2002. DOI : 10.1093/oso/9780198502845.001.0001.

[Poi16] Jérôme Poineau. "Raconte-moi... la droite de Berkovich". In : *Gazette de la S.M.F.* (2016). URL : https://smf.emath.fr/sites/default/files/2020-05/Poineau-site.pdf.

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