Talk 1: Dedekind Schemes

Lilly Steinbauer

October 13, 2026

Abstract

In this talk, we will introduce (or recall, depending on your knowledge) several different types of rings and schemes that will come up over the course of the seminar. We begin with integral schemes and recall some useful characterizations of such schemes. We then recall the dimension of a scheme and look at a few examples. Next, we consider normal schemes which we need to define Dedekind schemes, for which we will also discuss an example. Finally, we recall the definition of regularity for schemes and connect it to the previous concepts.

First, we recall what it means for a scheme to be irreducible, respectively reduced.

Definition 1 (Irreducible scheme; [AG1], 2.1)**.** A scheme X is *irreducible*, if its underlying topological space $|X|$ is irreducible, i. e. if $|X|$ cannot be written as a union of two closed proper subsets.

Lemma 2 ([Sta23], Tag 01OJ)**.** *Let X be a scheme. The following are equivalent:*

(a) X is irreducible.

(b) There exists an affine open covering $X = \bigcap_{i \in I} U_i$ such that U_i is irreducible for all $i \in I$ and $U_i \cap U_j \neq \emptyset$ for all $i, j \in I$.

(c) The scheme X is non-empty and every non-empty affine open $U \subseteq X$ is irreducible.

Definition 3 (Reduced scheme; [AG1], 2.3)**.** A scheme X is *reduced*, if the local ring $\mathcal{O}_{X,x}$ is a reduced ring for for every $x \in X$, i. e. $f \in \mathcal{O}_{X,x}$ with $f^n = 0$ already implies that $f = 0$.

Lemma 4 ([Sta23], Tag 01OJ)**.** *Let X be a scheme. The following are equivalent:*

(a) X is reduced.

(b) There exists an affine open covering $X = \bigcup_{i \in I} \operatorname{Spec} A_i$ such that each A_i is reduced.

(c) For every affine open subset $\operatorname{Spec} A \subseteq X$ the ring A is reduced.

Definition 5 (Integral scheme; [Sta23], Tag 01OJ)**.** A scheme X is *integral*, if it is non-empty and for every non-empty affine open $\operatorname{Spec} A \subseteq X$ the ring A is an integral domain.

Proposition 6 ([AG1], 2.4)**.** *Let X be a scheme. X is integral if and only if it is irreducible and reduced.*

Proof. If X is integral, then for any non-empty affine open $\operatorname{Spec} A \subseteq X$ the ring A is an integral domain. In particular, it is reduced (since every domain is reduced) and hence X is reduced by Lemma 4. We still need to check that X is irreducible. Suppose the contrary. Then there exist two non-empty open subsets $U_1, U_2 \subseteq X$ such that $U_1 \cap U_2 = \emptyset$. Choose non-empty affine open subschemes $\operatorname{Spec} A_i \subseteq U_i$ for $i = 1, 2$. Then $\operatorname{Spec} A_1 \cap \operatorname{Spec} A_2 = \emptyset$. Moreover, the disjoint union $\operatorname{Spec} A_1 \cup \operatorname{Spec} A_2 \cong \operatorname{Spec}(A_1 \times A_2)$ is a non-empty affine open subscheme of X. By assumption $A_1 \times A_2$ is therefore an integral domain; this is a contradiction. Thus, X is irreducible.

For the second direction, assume that X is irreducible and reduced. Let $\operatorname{Spec} A \subseteq X$ be a non-empty open affine subscheme. Since X is an irreducible scheme, it has a unique generic point η, which lies in every open subset, in particular $\eta \in \operatorname{Spec} A$. Since $\operatorname{Spec} A$ is irreducible and $\operatorname{Spec} A = V((0)) = V(\sqrt{(0)})$, the ideal $\sqrt{(0)}$ is prime and corresponds to η. Moreover, since X is reduced, so is A, and therefore $\sqrt{(0)} = \{a \in A \mid a^n = 0 \text{ for some } n \geq 1\} = (0)$. Hence, (0) is a prime ideal and so A is an integral domain. □

Remark 7. *Note that for integral schemes, we do not have equivalent conditions as in Lemma 2 and Lemma 4. If X is a scheme and $X = \bigcup_i \operatorname{Spec} A_i$ is an open affine covering of X with every A_i being an integral domain, it is in general not true that X is integral because it might not be irreducible.*

We now recall the definition of dimension for schemes:

Definition 8 (Dimension of a scheme; [CA], 1.4.1)**.** Let X be a topological space. Then we define the *(Krull) dimension of X* to be the length of the longest strictly ascending chain of irreducible and closed subsets of X, i. e.

$$\begin{aligned}\dim X &= \sup\{l \in \mathbb{N} \mid \exists Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_l \subseteq X \text{ with } Z_i \subseteq X \text{ closed and irreducible}\} \\ &= \sup\{\dim Z \mid Z \text{ is an irreducible component of } X\}.\end{aligned}$$

Now, let X be a scheme. Then the *dimension of X* is defined to be the dimension of the underlying topological space $|X|$.

Lemma 9 ([CA])**.** *For an affine scheme $X = \operatorname{Spec} A$, we have $\dim X = \dim A$.*

Proof. This follows immediately from the following bijection:

$$\begin{array}{ccc} \{Z \subseteq X \text{ irreducible and closed}\} & \longleftrightarrow & \{\mathfrak{p} \subseteq A \text{ prime ideals}\} \\ Z & \longmapsto & \text{generic point } \eta \text{ of } Z \\ \overline{\{\mathfrak{p}\}} = V(\mathfrak{p}) & \longleftarrow\!\shortmid & \mathfrak{p} \end{array}$$

□

Example 10. Consider the scheme $\mathbb{A}_k^n$. By the previous lemma, it is of dimension n as we know that the polynomial ring $k[x_1, ..., x_n]$ is of dimension n and $\mathbb{A}_k^n = \operatorname{Spec} k[x_1, ...x_n]$. In the polynomial ring $k[x_1, ...x_n]$ we have a maximal chain of prime ideals given by $(0) \subsetneq (x_1) \subsetneq ... \subsetneq (x_1, ..., x_n)$. The details for this example can be found in [CA], Theorem 1.4.11.

Example 11. Let $X := \operatorname{Spec} A$ with $A := k[x, y]/(xy)$, where k is algebraically closed. By the correspondence between prime ideals in the quotient ring A and prime ideals in $k[x, y]$ containing (xy), we can write $X \cong V(xy) \subseteq \mathbb{A}_k^2$. Since $V(xy) = V(x) \cup V(y)$, X is geometrically just the union of the two coordinate axes, hence a "cross".

Let $\overline{x}$ and $\overline{y}$ denote the residue classes of x and y in the ring A. $V(\overline{x})$ and $V(\overline{y})$ are irreducible subsets of X because $A/(\overline{x}) \cong k[y]$, respectively $A/(\overline{y}) \cong k[x]$, are integral domains.

Moreover, $V(\overline{x})$ and $V(\overline{y})$ are even irreducible components of X: Since irreducible components of X correspond to minimal primes of A, we have to show that $(\overline{x})$ and $(\overline{y})$ are indeed minimal prime ideals. Since we have $\overline{x} \cdot \overline{y} = 0$ in A, every prime ideal $\mathfrak{p} \subseteq A$ contains $\overline{x} \cdot \overline{y}$ and thus contains $\overline{x}$ or $\overline{y}$, i. e. $(\overline{x}) \subseteq \mathfrak{p}$ or $(\overline{y}) \subseteq \mathfrak{p}$. But both $(\overline{x})$ and $(\overline{y})$ are themselves prime and neither one contains the other. Therefore, they are the unique minimal primes of A and correspondingly, $V(\overline{x})$ and $V(\overline{y})$ are the irreducible components of X. Additionally, $V(\overline{x}) \cong \operatorname{Spec} k[y] \cong \mathbb{A}_k^1 \cong \operatorname{Spec} k[x] \cong V(\overline{y})$, hence $\dim V(\overline{x}) = \dim V(\overline{y}) = 1$. By the definition of the dimension, this gives $\dim X = 1$.

Example 12. Consider the scheme $X := \operatorname{Spec} \mathbb{Z}$. The only prime ideals in $\mathbb{Z}$ are (0) and (p) for any prime number p. Since $\mathbb{Z}/p\mathbb{Z} \cong \mathbb{F}_p$, the ideal (p) is even a maximal ideal of the ring $\mathbb{Z}$ for any prime number p. This means that $(0) \subsetneq (p)$ is already a maximal chain of prime ideals in $\mathbb{Z}$ and hence, $\dim X = \dim \mathbb{Z} = 1$.

Now we want to define normal schemes. We will need this type of scheme for the definition of Dedekind schemes, but we will also encounter the notion throughout the seminar when talking about models:

Consider $\operatorname{Spec} A$, where $A := \mathbb{Z}[y, z]/(4y - z^2)$. Roughly, this is $\mathbb{Z}[x^2, 2x] \subseteq \mathbb{Z}[x]$ by mapping $y \mapsto x^2$ and $z \mapsto 2x$. Over $V(2)$, we have $R[1/2] = \mathbb{Z}[x]$ (indeed $y = (z/2)^2$). This is a model of $\mathbb{A}_{\mathbb{Q}}^1$. However, it is not normal: In its fraction field $\mathbb{Q}(x)$, the element

x is a root of $X^2 - x^2$. Put differently, the element $z/2$ is a root of $X^2 - y$ (one has to check that $z/2$ does not exist in R).

Definition 13 (Normal ring; [Sta23], Tag 037B)**.** An integral domain A is said to be *normal*, if it is integrally closed in its field of fractions. That means that if $a \in \operatorname{Quot} A$ and a is integral over A, then $a \in A$.

More generally, an arbitrary commutative ring A is called *normal* if for each prime ideal $\mathfrak{p} \subseteq A$, the localization $A_\mathfrak{p}$ is a normal domain.

Definition 14 (Normal scheme; [Sta23], Tag 033H)**.** A scheme X is *normal*, if the local ring $\mathcal{O}_{X,x}$ is a normal integral domain for every $x \in X$.

Lemma 15 ([Sta23], Tag 033H)**.** *Let X be a scheme. Then the following are equivalent:*

(a) X is normal.

(b) For every affine open $\operatorname{Spec} A \subseteq X$, the ring A is normal.

(c) There exists an affine open covering $X = \bigcup_i U_i$, with $U_i = \operatorname{Spec} A_i$, such that A_i is a normal ring for every i.

Moreover, if X is normal then every open subscheme of X is normal too.

Proof. For the first direction, let X be normal and let $U = \operatorname{Spec} A \subseteq X$ be an affine open subset. Take any prime ideal $\mathfrak{p} \in \operatorname{Spec} A$ and let x be the corresponding point in U. Then $A_\mathfrak{p} \cong \mathcal{O}_{U,x} = \mathcal{O}_{X,x}$. Since X is normal, $\mathcal{O}_{X,x}$ is normal, hence $A_\mathfrak{p}$ is normal. Therefore, A is normal.

The proof from (b) to (c) is clear. Now suppose that (c) holds, i. e. write $X = \bigcup_i U_i$ with $U_i = \operatorname{Spec} A_i$ being open and affine and suppose that each A_i is normal. Take any $x \in X$ and choose $i \in I$ such that $x \in U_i$. Let $\mathfrak{p} \subseteq A_i$ be the prime ideal corresponding to x. Then $\mathcal{O}_{X,x} = \mathcal{O}_{U_i,x} \cong (A_i)_\mathfrak{p}$. Since A_i is normal, every localization $(A_i)_\mathfrak{p}$ is normal and hence $\mathcal{O}_{X,x}$ is normal. Since x was arbitrary, X is already a normal scheme, which finishes the proof. □

We want to give another characterization of normal schemes. For this, we first need an analogous statement about rings:

Lemma 16 ([Sta23], Tag 030C)**.** *Let A be a ring. Assume that A is reduced and has finitely many minimal primes. Then the following are equivalent:*

(a) A is a normal ring.

(b) A is a finite product of normal domains.

Lemma 17 ([Sta23], Tag 0357)**.** *Let X be a scheme such that any quasi-compact open has a finite number of irreducible components. Then the following are equivalent:*

(a) X is normal.

(b) X is a disjoint union of normal integral schemes.

Proof. To get from (b) to (a), suppose that $X = \bigcup_i X_i$, where each X_i is integral and normal. Let $x \in X$. Then there exists some i such that $x \in X_i$. Since X_i is open, we have $\mathcal{O}_{X,x} \cong \mathcal{O}_{X_i,x}$. By normality of X_i, $\mathcal{O}_{X_i,x}$ is a normal domain, hence $\mathcal{O}_{X,x}$ is one too. As x was arbitrary, this shows that X is normal.

Suppose (a) holds. We first consider the case where $X = \operatorname{Spec} A$ is affine. Then A is normal, and hence reduced. Since X is quasi-compact, A has only finitely many minimal primes. Then by the previous lemma, there exist $A_1, ..., A_n$ such that $A \cong \prod_{i=1}^{n} A_i$, which means we can write $X = \coprod_{i=1}^{n} \operatorname{Spec} A_i$. Since the A_i are normal and integral, the $\operatorname{Spec} A_i$ are so too. This proves the proposition for the affine case but we want to say something about the irreducible components of X in this case, which we will use in the general case: The $\operatorname{Spec} A_i$ are precisely the irreducible components of X. Each of them is irreducible since they are integral. Also we have a decomposition into pairwise disjoint open open and closed subsets. This means that every irreducible subset is contained in one of the $\operatorname{Spec} A_i$.

For the general case, let X be a not necessarily affine scheme and let $\bigcup_i U_i = X$ be an affine open covering. Then U_i is normal (by Lemma 15 and quasi-compact for each i. From the affine case we know that the irreducible components of U_i are open, integral and normal. Now, let $Z \subseteq X$ be an irreducible component of X. Then for each i such that $Z \cap U_i \neq \emptyset$, $Z \cap U_i$ is an irreducible component of U_i. This means that $Z \cap U_i$ must be open in U_i, hence also in X. Thus, $Z = \bigcup_i (Z \cap U_i)$ is open in X. Since Z is open, and X is normal by assumption, Z itself is also normal, and hence reduced. This gives us that Z is integral. What is left to show is that the irreducible components of X are disjoint. Suppose otherwise: Let $Z_1 \cap Z_2 \neq \emptyset$ be two distinct irreducible components. Since Z_2 is open, $Z_1 \cap Z_2$ is an open subset of Z_1. Since Z_1 is irreducible, $\overline{Z_1 \cap Z_2} = Z_1$. Now Z_2 is also closed in X, hence $Z_1 = \overline{Z_1 \cap Z_2} \subseteq Z_2$, which is a contradiction to the maximality of irreducible components. Thus, (b) follows. □

Example 18. We look at $X := \operatorname{Spec} \mathbb{Z}$ again. The ring $\mathbb{Z}$ is an integral domain and it is integrally closed in its fraction field $\operatorname{Quot} \mathbb{Z} = \mathbb{Q}$. Hence $\mathbb{Z}$ is a normal domain. Normality is preserved under localization, so for every $\mathfrak{p} \in \operatorname{Spec} \mathbb{Z}$, the local ring $\mathcal{O}_{X,\mathfrak{p}} \cong \mathbb{Z}_{\mathfrak{p}}$ is normal. Moreover, X is irreducible because $\mathbb{Z}$ is an integral domain. Thus, X is a normal scheme.

We have already shown in Example 12 that $\operatorname{Spec}\mathbb{Z}$ is of dimension one. Now we have also seen that it is a normal scheme. Moreover, $\mathbb{Z}$ is a Noetherian integral domain, which means that $\operatorname{Spec}\mathbb{Z}$ is a Noetherian scheme. In particular, it is locally Noetherian. This leads to the definition of yet another type of scheme, Dedekind schemes, for which $\operatorname{Spec}\mathbb{Z}$ is an example.

Definition 19 (Dedekind scheme; [LE02], 4.1.2)**.** A *Dedekind domain* is a normal Noetherian integral domain of dimension 0 or 1. A *Dedekind scheme* is a normal locally Noetherian scheme of dimension 0 or 1.

Example 20. As we have seen, $\operatorname{Spec}\mathbb{Z}$ is a Dedekind scheme.

Example 21. More generally, if A is a Dedekind domain, then $\operatorname{Spec}A$ is a Dedekind scheme.

If X is a Noetherian integral scheme, then X is a Dedekind scheme if and only if $\mathcal{O}_X(U)$ is a Dedekind domain for every non-empty open subset $U \subseteq X$.

Definition 22 (Regular scheme; [AG2], 1.1.2)**.** A scheme X is *regular*, if it is locally Noetherian $\mathcal{O}_{X,x}$ is a regular local ring for every point $x \in X$.

Lemma 23 ([Sta23], Tag 02IR)**.** *Let X be a scheme. Then the following are equivalent:*

(a) X is regular.

(b) X is locally Noetherian and for every closed point $x \in X$, $\mathcal{O}_{X,x}$ is regular.

(c) For each $x \in X$, there exists an affine open neighbourhood $U \subseteq X$ of x such that $\mathcal{O}_X(U)$ is Noetherian and regular.

Example 24. Again we look at $X := \operatorname{Spec}\mathbb{Z}$. We already know that X is locally Noetherian. What we still need to show is that $\mathcal{O}_{X,x} = \mathbb{Z}_x$ is regular for each $x \in X$. There are only two types of points in X: The generic point (0) and the closed points (p) for a prime number p. For $x = (0)$, we have $\mathcal{O}_{X,x} = \mathbb{Z}_{(0)} = \mathbb{Q}$, which is a field and hence regular. For $x = (p)$, we have $\mathcal{O}_{X,x} = \mathbb{Z}_{(p)}$. This is a local Noetherian domain with maximal ideal $\mathfrak{m} = p\mathbb{Z}_{(p)}$, which is a principal ideal. Thus, $\mathbb{Z}_{(p)}$ is a DVR and hence regular. This shows that X is a regular scheme.

In fact, every Dedekind scheme is regular because each of its local rings is either a field (for the generic point) or a DVR (for the closed points). The proof is omitted.

For our final proposition, we need a ring-theoretic lemma. The proof can be found in Hideyuki Matsumura's *Commutative Ring Theory* (1986).

Lemma 25 ([Mat86], 11.2)**.** *Let A be a ring. Then the following are equivalent:*

(a) A is a discrete valuation ring[1]*.*

[1]Here I am using the definition from [AG2], Definition 1.1.3

(b) A is a one-dimensional normal Noetherian local ring

Proposition 26. *Let C be a normal curve over a field k. Then C is regular.*

Proof. Take an arbitrary closed point $P \in |C|$. Then $\mathcal{O}_{C,P}$ is Noetherian because C is Noetherian. It is also normal because C is normal, and hence all its local rings are normal rings. Additionally $\mathcal{O}_{C,P}$ is one-dimensional because C is an integral scheme of dimension one and P is a closed point (see [AG2], Consequence 1.1.10). By Lemma 25, this gives us that $\mathcal{O}_{C,P}$ is a discrete valuation ring, and hence regular.

At the generic point $\eta \in C$, we have $\mathcal{O}_{C,\eta} = k(C)$, which is a field and thus also local and regular. This means that $\mathcal{O}_{C,x}$ is a regular local ring for every $x \in C$, which is exactly what it means for a scheme to be regular. □

References

[CA] Urs Hartl. *Commutative Algebra*. Lecture given at Universität Münster (Autumn 2024).

[AG1] Urs Hartl. *Algebraic Geometry I*. Lecture given at Universität Münster (Autumn 2025).

[AG2] Urs Hartl. *Algebraic Geometry II*. Lecture given at Universität Münster (Summer 2026).

[LE02] Qing Liu, Reinie Erné. *Algebraic Geometry and Arithmetic Curves*. Oxford University Press (2002).

[Mat86] Hideyuki Matsumura. *Commutative Ring Theory*. Cambridge University Press (1986).

[Sta23] The Stacks Project Authors. *The Stacks Project*. `https://stacks.math.columbia.edu` (2023).