Projective geometry and smoothness

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1 Proj of a graded ring

Definition 1.1. Let $S = \bigoplus_{d\geq 0} S_d$ be a graded ring with irrelevant ideal $S_+ = \bigoplus_{d>0} S_d$. Then

$$\operatorname{Proj}(S) = \{\, \mathfrak{p} \subset S \text{ homogeneous prime} : S_+ \not\subseteq \mathfrak{p} \,\},$$

with closed sets $V_+(I)$ for homogeneous ideals I, and basic opens

$$D_+(f) \;\cong\; \operatorname{Spec}\big(S_{(f)}\big), \qquad S_{(f)} := (S_f)_0,$$

for f homogeneous of degree > 0.

Remark. Suppose S_+ is generated, as an ideal, by homogeneous elements f_λ of positive degree. Then

$$\operatorname{Proj}(S) = \bigcup_{\lambda} D_+(f_\lambda).$$

Indeed, let $\mathfrak{p} \in \operatorname{Proj}(S)$. If $\mathfrak{p}$ contained every f_λ it would contain the ideal they generate, namely S_+, which is excluded by the definition of Proj. So $f_\lambda \notin \mathfrak{p}$ for some λ, i.e. $\mathfrak{p} \in D_+(f_\lambda)$. Conversely every $D_+(f_\lambda)$ lies in $\operatorname{Proj}(S)$ by definition.

2 Projective space

Definition 2.1. $\mathbb{P}^n_{\mathbb{Z}} = \operatorname{Proj} \mathbb{Z}[T_0, \ldots, T_n]$ with $\deg T_i = 1$. For a scheme S, the base change $\mathbb{P}^n_S = \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} S$ is projective n-space over S; over a ring, $\mathbb{P}^n_R = \operatorname{Proj} R[T_0, \ldots, T_n]$.

Lemma 2.2. *$\mathbb{P}^n_{\mathbb{Z}} = \bigcup_{i=0}^n D_+(T_i)$, and $D_+(T_i) \cong \operatorname{Spec}\mathbb{Z}[T_0/T_i, \ldots, \widehat{T_i/T_i}, \ldots, T_n/T_i] \cong \mathbb{A}^n_{\mathbb{Z}}$. The same holds after base change to any ring R, with $\mathbb{Z}$ replaced by R throughout, since $\mathbb{Z}[T_j/T_i] \otimes_{\mathbb{Z}} R = R[T_j/T_i]$, which is also why $\mathbb{P}^n_R = \operatorname{Proj} R[T_0, \ldots, T_n]$. If $R = k$ is an algebraically closed field, the closed points of $\mathbb{P}^n_k$ are the lines through the origin in k^{n+1}, written $[a_0 : \cdots : a_n]$ with the a_i not all zero and taken up to common scaling.*

Lemma 2.3. *$\mathbb{P}^n_S \to S$ is universally closed and satisfies the existence and uniqueness parts of the valuative criterion. It is also separated and quasi-compact.*

3 The twisting sheaves $\mathcal{O}(n)$

Definition 3.1. Let $X = \operatorname{Proj}(S)$. The n-th twist of the structure sheaf is

$$\mathcal{O}_X(n) := \widetilde{S(n)}, \qquad S(n)_d = S_{n+d},$$

and for an $\mathcal{O}_X$-module $\mathcal{F}$ we set $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

Definition 3.2. From the identification $S(n) \otimes_S S(m) = S(n+m)$ one obtains canonical maps of $\mathcal{O}_X$-modules

$$\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) \longrightarrow \mathcal{O}_X(n+m), \qquad \mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{F}(m) \longrightarrow \mathcal{F}(n+m),$$

and, on global sections, a canonical map of graded rings

$$S \longrightarrow \bigoplus_{n \geq 0} \Gamma\big(X, \mathcal{O}_X(n)\big).$$

Lemma 3.3. *Let $f \in S$ be homogeneous of degree $d > 0$. On $D_+(f)$, multiplication by f*

$$S_{(f)} \xrightarrow{\cdot f} S(d)_{(f)}, \qquad \tfrac{g}{f^m} \longmapsto \tfrac{fg}{f^m},$$

is an isomorphism, since f is a unit there. Hence $\mathcal{O}_X(nd)|_{D_+(f)}$ is trivial for all $n \in \mathbb{Z}$, with generator f^n.

Lemma 3.4. *Assume S is generated by S_1 over S_0, equivalently that X is covered by $D_+(f)$ with $f \in S_1$; e.g. $S = k[T_0, \ldots, T_N]$. Then all $\mathcal{O}_X(n)$ are invertible, the maps of Definition 3.2 are isomorphisms, and*

$$\mathcal{O}_X(1)^{\otimes n} \cong \mathcal{O}_X(n), \qquad \widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) \cong \widetilde{M(n)}.$$

4 Maps to projective space

Lemma 4.1. *$\mathbb{P}^n_{\mathbb{Z}}$ represents the functor sending a scheme Y to the set of pairs $\big(\mathcal{L}, (s_0, \ldots, s_n)\big)$, where $\mathcal{L}$ is an invertible $\mathcal{O}_Y$-module and $s_0, \ldots, s_n \in \Gamma(Y, \mathcal{L})$ generate $\mathcal{L}$, two pairs $\big(\mathcal{L}, (s_i)\big)$ and $\big(\mathcal{N}, (t_i)\big)$ being identified when there is an isomorphism $\beta : \mathcal{L} \to \mathcal{N}$ with $\beta(s_i) = t_i$ for every i. Equivalently, a morphism $\varphi : Y \to \mathbb{P}^n_{\mathbb{Z}}$ is exactly the data*

$$\mathcal{L} = \varphi^* \mathcal{O}(1), \qquad s_i = \varphi^* T_i.$$

Remark. The sections $s_0, \ldots, s_n$ *generate* $\mathcal{L}$ if for every $y \in Y$ their images in the fibre $\mathcal{L} \otimes \kappa(y)$ span it over $\kappa(y)$; equivalently, the induced map $\mathcal{O}_Y^{n+1} \to \mathcal{L}$ is surjective.

Remark. For $y \in Y$ write $s_i(y)$ for the image of s_i in the fibre $\mathcal{L} \otimes \kappa(y)$, and set

$$Y_{s_i} = \{\, y \in Y \;:\; s_i(y) \neq 0 \,\},$$

an open subset of Y. Generation says precisely that $Y = \bigcup_i Y_{s_i}$: the fibre $\mathcal{L} \otimes \kappa(y)$ is a one-dimensional $\kappa(y)$-vector space, so the $s_i(y)$ span it exactly when at least one of them is nonzero, that is, exactly when y lies in some Y_{s_i}.

On Y_{s_i} the section s_i generates $\mathcal{L}$, so for each j there is a unique $g_{ij} \in \mathcal{O}_Y(Y_{s_i})$ with $s_j = g_{ij} s_i$; one writes $s_j / s_i := g_{ij}$. These ratios are the candidate coordinates of the map, just as the chart coordinates T_j/T_i on $\mathbb{P}^n$ are themselves ratios of the sections T_i of $\mathcal{O}(1)$.

To turn them into a morphism, recall that for a scheme Z and a ring A one has $\mathrm{Hom}(Z, \operatorname{Spec} A) = \mathrm{Hom}_{\mathrm{ring}}\big(A, \mathcal{O}_Z(Z)\big)$, and that a ring homomorphism out of a polynomial ring is freely determined by the images of the variables. Hence

$$\mathrm{Hom}\big(Y_{s_i}, D_+(T_i)\big) = \mathrm{Hom}_{\mathrm{ring}}\big(\mathbb{Z}[T_0/T_i, \ldots, \widehat{T_i/T_i}, \ldots, T_n/T_i],\, \mathcal{O}_Y(Y_{s_i})\big) = \mathcal{O}_Y(Y_{s_i})^{\,n},$$

and the tuple $(g_{ij})_{j \neq i}$ defines a morphism $\varphi_i : Y_{s_i} \to D_+(T_i) \subseteq \mathbb{P}^n$.

Uniqueness of the g_{ij} forces $g_{ik} = g_{jk}\, g_{ij}$, which is exactly the relation $T_k/T_i = (T_k/T_j)(T_j/T_i)$ among the coordinates of $\mathbb{P}^n$. So the φ_i agree on overlaps and glue to a single $\varphi : Y \to \mathbb{P}^n$.

On Y_{s_i} both $\varphi^*\mathcal{O}(1)$ and $\mathcal{L}$ are free of rank one, on φ^*T_i and on s_i respectively, so sending the one generator to the other defines an isomorphism $\varphi^*\mathcal{O}(1) \cong \mathcal{L}$. It carries φ^*T_j to s_j for every j: on $D_+(T_i)$ one has $T_j = (T_j/T_i)\,T_i$, and φ sends the function T_j/T_i to s_j/s_i, so $\varphi^*T_j = (s_j/s_i)\,\varphi^*T_i$, which goes to $(s_j/s_i)\,s_i = s_j$. Hence $\varphi^*T_i = s_i$ for all i.

5 Closed subschemes of $\mathbb{P}^n$

Proposition 5.1. *Let $Z \subset \mathbb{P}^n_R$ be closed. For $d \geq 0$ set*

$$I_d = \ker\big(R[T_0,\dots,T_n]_d \longrightarrow \Gamma(Z, \mathcal{O}(d)|_Z)\big),$$

the degree-d forms vanishing on Z. Then $I = \bigoplus_d I_d$ is a homogeneous ideal and $Z = \operatorname{Proj}\big(R[T]/I\big)$.

Example 5.2. Take $R = k$, so that $\mathbb{P}^1_k = \operatorname{Proj} k[T_0, T_1]$, and let $Z = \{[0:1]\} \subset \mathbb{P}^1_k$ be the point where T_0 vanishes. Since Z is a single point, $\Gamma(Z, \mathcal{O}(d)|_Z)$ is a 1-dimensional k-vector space, generated by the restriction of $T_1^{\,d}$ because T_1 does not vanish on Z. The map of the proposition is therefore

$$k[T_0,T_1]_d \longrightarrow \Gamma\big(Z, \mathcal{O}(d)|_Z\big) \cong k, \qquad F = \sum_{\ell=0}^{d} a_\ell T_0^{\ell} T_1^{d-\ell} \longmapsto a_0,$$

since every term with $\ell \geq 1$ is a multiple of T_0 and so restricts to 0 on Z. Its kernel is thus the forms with $a_0 = 0$, that is, those divisible by T_0:

$$I_0 = 0, \qquad I_1 = k\,T_0, \qquad I_2 = k\,T_0^2 + k\,T_0T_1,$$
$$I_3 = k\,T_0^3 + k\,T_0^2T_1 + k\,T_0T_1^2, \qquad \dots, \qquad I_d = T_0 \cdot k[T_0,T_1]_{d-1}.$$

and $I = \bigoplus_d I_d = (T_0)$. The proposition then returns $\operatorname{Proj}\big(k[T_0,T_1]/(T_0)\big) = \operatorname{Proj} k[T_1]$, a single point, namely Z.

6 Projective morphisms

Definition 6.1. Let $f : X \to S$ be a morphism of schemes. We say f is *projective* if X is isomorphic as an S-scheme to a closed subscheme of $\mathbb{P}(\mathcal{E})$ for some quasi-coherent finite type $\mathcal{O}_S$-module $\mathcal{E}$, and *locally projective* if S has an open covering $S = \bigcup U_i$ with each $f^{-1}(U_i) \to U_i$ projective.

Remark. $\mathbb{P}(\mathcal{E}) = \operatorname{Proj}_S(\operatorname{Sym}\mathcal{E})$, where $\mathcal{E}$ is only assumed quasi-coherent of finite type. Locally on S this is a closed subscheme of a projective space, and it is all of $\mathbb{P}^n_U$ precisely when $\mathcal{E}$ is free of rank $n+1$ there; for $\mathcal{E} = \mathcal{O}_S^{\,n+1}$ one recovers $\mathbb{P}^n_S$.

Lemma 6.2. *A locally projective morphism is proper, i.e. separated, of finite type, and universally closed.*

7 Flatness and the two fibres

Definition 7.1. M is flat over R if $- \otimes_R M$ preserves injections. Over a discrete valuation ring, flat $\iff$ torsion-free.

Definition 7.2. For X over $\mathcal{O}_K$ with uniformizer t, fraction field K and residue field k:

$$X_K = X \times_{\mathcal{O}_K} \operatorname{Spec} K \quad \text{(generic fibre)}, \qquad X_k = X \times_{\mathcal{O}_K} \operatorname{Spec} k \quad \text{(special fibre)},$$

and on $X = \operatorname{Spec} A$ these are $X_K = \operatorname{Spec} A[1/t]$ and $X_k = \operatorname{Spec} A/tA$.

Example 7.3. Let $A = \mathbb{Z}_p[x,y]/(xy-p)$. Since $\mathbb{Q}_p = \mathbb{Z}_p[1/p]$ and $\mathbb{F}_p = \mathbb{Z}_p/(p)$, the two fibres are

$$A \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = A[1/p] = \mathbb{Q}_p[x,y]/(xy-p),$$

$$A \otimes_{\mathbb{Z}_p} \mathbb{F}_p = A/pA = \mathbb{F}_p[x,y]/(xy-p) = \mathbb{F}_p[x,y]/(xy),$$

the last equality is because $p = 0$ in $\mathbb{F}_p$. So the generic fibre is $xy = p$ over $\mathbb{Q}_p$ and the special fibre is $xy = 0$ over $\mathbb{F}_p$.

8 Arithmetic surfaces

Definition 8.1. Let S be a Dedekind scheme of dimension 1. An *arithmetic surface* over S is an integral, projective, flat S-scheme $X \to S$ of dimension 2. It is a *normal* arithmetic surface if moreover X is normal, and a *regular* arithmetic surface if moreover X is regular.

Example 8.2. $\mathbb{P}^1_{\mathcal{O}_K}$ is an arithmetic surface over $\operatorname{Spec} \mathcal{O}_K$. But $\mathbb{P}^1_k$ is not one over $\operatorname{Spec} k$, and $\mathbb{P}^n_{\mathcal{O}_K}$ is not one for $n \geq 2$.

9 A normality criterion

Lemma 9.1. *Let $\mathcal{O}_K$ be a discrete valuation ring with fraction field K and residue field k. Let X be an $\mathcal{O}_K$-scheme such that $\mathcal{O}_X(U)$ is flat over $\mathcal{O}_K$ for every affine open $U \subseteq X$. Suppose X_K is normal and X_k is reduced. Then X is normal.*

Proof. Normality is local, so assume $X = \operatorname{Spec} A$ affine, and A a domain (in the application X is integral by definition). By flatness t is a nonzerodivisor, so $A \subseteq A[1/t] \subseteq \operatorname{Frac}(A)$.

Let $\alpha \in \operatorname{Frac}(A)$ be integral over A. It is then integral over $A[1/t]$, which is integrally closed since X_K is normal, so $\alpha \in A[1/t]$ and

$$\alpha = t^{-r} a, \qquad a \in A,\ r \in \mathbb{Z}.$$

If $r \leq 0$ then $\alpha \in A$ and we are done, so suppose $r > 0$. If $a \in tA$, write $a = ta_1$ and replace (r, a) by $(r-1, a_1)$; since r decreases this terminates, so we may assume $a \notin tA$.

Take an integral relation $\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_0 = 0$ with $a_i \in A$ and multiply by t^{rn}. Using $t^{rn}\alpha^i = t^{r(n-i)}a^i$,

$$a^n = -\sum_{i<n} a_i\, t^{\,r(n-i)} a^i.$$

For $r > 0$ each exponent $r(n-i) \geq 1$, so the right-hand side lies in tA and $a^n \in tA$; that is, $\bar{a}$ is nilpotent in $A/tA = \mathcal{O}(X_k)$. Since X_k is reduced, $\bar{a} = 0$, i.e. $a \in tA$, contradicting the normalization.

Hence $r \leq 0$ and $\alpha \in A$, so A is integrally closed. □

Example 9.2. Let $A = \mathbb{Z}_p[x,y]/(y^2 - p^2x)$. The two fibres are

$$A \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = A[1/p] = \mathbb{Q}_p[x,y]/(y^2 - p^2x),$$

$$A \otimes_{\mathbb{Z}_p} \mathbb{F}_p = A/pA = \mathbb{F}_p[x,y]/(y^2 - p^2x) = \mathbb{F}_p[x,y]/(y^2),$$

the last step because $p^2 = 0$ in $\mathbb{F}_p$.

The generic fibre is normal: over $\mathbb{Q}_p$ the substitution $y = pz$ is invertible and turns $y^2 = p^2x$ into $z^2 = x$, so $A[1/p] \cong \mathbb{Q}_p[x,z]/(z^2 - x) \cong \mathbb{Q}_p[z]$, a polynomial ring and hence normal. The special fibre is $\mathbb{F}_p[x,y]/(y^2)$, which is *not* reduced: $y \neq 0$ but $y^2 = 0$.

And A is not normal. In $\mathrm{Frac}(A)$ put $\alpha = y/p$; then

$$\alpha^2 = \frac{y^2}{p^2} = \frac{p^2x}{p^2} = x \in A,$$

so α satisfies the integral relation $\alpha^2 - x = 0$, yet $\alpha \notin A$. This is the proof of the lemma running in reverse: here $r = 1$ and $a = y \notin pA$, with y nilpotent modulo p , precisely the situation reducedness of the special fibre was used to exclude.

10 Smooth morphisms

Definition 10.1. Let R be a ring, $n \geq c \geq 0$, and $f_1, \ldots, f_c \in R[x_1, \ldots, x_n]$. Then $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is standard smooth over R if

$$g = \det \left(\frac{\partial f_i}{\partial x_j} \right)_{1 \leq i,j \leq c}$$

is invertible in S. A ring map $R \to S$ is standard smooth if S is R-isomorphic to such an algebra.

Example 10.2. In $R = k[x]$ the element x is nonzero but not invertible: it vanishes at the origin. In the localisation $k[x, x^{-1}]$, the functions on $\mathbb{A}^1 \smallsetminus \{0\}$, it is invertible, the point where it vanished having been removed. So "$g \neq 0$" says only that g does not vanish identically, whereas "g invertible" says that it vanishes nowhere on $\operatorname{Spec} S$. In a field the two coincide .

Example 10.3. For $c = 0$ there are no equations and $S = R[x_1, \ldots, x_n]$. The determinant of the empty 0×0 matrix is 1, so $\mathbb{A}^n_R \to \operatorname{Spec} R$ is standard smooth of relative dimension n.

Example 10.4. Take $n = 3$, $c = 2$ and $f_1 = y - x^2$, $f_2 = z - x^3$ in $R[x,y,z]$, so the expected relative dimension is $3 - 2 = 1$. The Jacobian is

$$J = \begin{pmatrix} -2x & 1 & 0 \\ -3x^2 & 0 & 1 \end{pmatrix}.$$

The minor on the first two columns is $3x^2$, not invertible; the minor on the last two is the identity, of determinant 1. Reordering the variables as (y, z, x) therefore exhibits the algebra as standard smooth of relative dimension 1. The equations solve for y and z in terms of x, and the free variable is exactly the column left out.

Example 10.5. Let $S = \mathbb{R}[x,y]/(x^2 + y^2 - 1)$, so $n = 2$, $c = 1$ and the Jacobian is $(2x, 2y)$. The minor on the first column is $g = 2x$, which is nonzero in S but not invertible, vanishing at $(0, \pm 1)$. On the locus $x \neq 0$ it is invertible, and on $y \neq 0$ one uses the other minor $2y$; the two loci cover the circle. Which variable can be solved for depends on the point, which is why the definition of a smooth morphism must be local.

Definition 10.6. $f : X \to S$ is smooth at x if there are affine opens $x \in U \subseteq X$ and $f(U) \subseteq V \subseteq S$ with $f|_U : U \to V$ standard smooth; f is smooth if it is smooth at every point. That is, f is smooth if and only if it is locally standard smooth.

Example 10.7. For $R[x,y]/(y^2 - x^3)$ the Jacobian is $(-3x^2, 2y)$ and neither entry is invertible, both vanishing at the origin. Unlike the circle, no change of chart helps: the origin is a genuine singularity.

Lemma 10.8. *A morphism f is smooth if and only if it is flat, locally of finite presentation, and all its fibres X_s are smooth.*

Remark. Let $f : X \to S$ be a morphism of schemes, and for the computation take $S = \operatorname{Spec} R$ affine and $X = \operatorname{Spec} A$ with

$$A = R[x_1, \dots, x_n]/(f_1, \dots, f_c),$$

the morphism being the one induced by $R \hookrightarrow A$.

Smoothness was defined above by equations and determinants. There is an equivalent formulation in terms of the sheaf of relative differentials $\Omega^1_{X/S}$, which is what makes it computable over a field. Since limits are unavailable, d is defined by the rules it ought to satisfy: $\Omega^1_{X/S}$ is generated by symbols dg, one for each $g \in \mathcal{O}_X$, subject to additivity $d(g+h) = dg + dh$, the Leibniz rule $d(gh) = g\,dh + h\,dg$, and $dr = 0$ for every r coming from the base R — the analogue of $d(5) = 0$, with "constant" now meaning "coming from S".

On a polynomial ring these rules reduce every dg to a combination of the dx_j. Imposing $f_i = 0$ forces $df_i = \sum_j (\partial f_i/\partial x_j)\,dx_j = 0$, one relation per equation, whose coefficients form the i-th row of the Jacobian. Hence

$$\Omega^1_{X/R} = \Big(\bigoplus_j \mathcal{O}_X\,dx_j\Big)\Big/\Big(\sum_j \frac{\partial f_i}{\partial x_j}\,dx_j \;:\; i = 1, \dots, c\Big) = \operatorname{coker}(J).$$

The c relations cut the rank from n down to $n - c$ exactly when they are independent, i.e. when J has full rank. So "Ω^1 free of rank $n-c$" and "the Jacobian has full rank c" are the same statement.

Remark. Smoothness was defined above by the Jacobian condition. There is another definition in the literature, by base change to the algebraic closure: X is *smooth at x* if the points of $X_{\bar{k}}$ lying above x are regular points of $X_{\bar{k}}$, and X is smooth over k if this holds everywhere, i.e. if $X_{\bar{k}}$ is regular.

Proposition 10.9. *Let X be an algebraic variety over a field k and $x \in X$. The following are equivalent:*

(i) X is smooth in a neighbourhood of x;

(ii) $\Omega^1_{X,x}$ is free of rank $\dim_x X$;

(iii) X is smooth at x.

Lemma 10.10. *Let k be a field, S a finite type k-algebra and $\mathfrak{q} \subset S$ a prime. Assume $\kappa(\mathfrak{q})$ is separable over k. Then S is smooth over k at $\mathfrak{q}$ if and only if the local ring $S_{\mathfrak{q}}$ is regular.*

Remark. A ring map $R \to S$ is *formally smooth* if for every commutative solid diagram

$$\begin{array}{ccc} S & \longrightarrow & A/I \\ \uparrow & \nearrow & \uparrow \\ R & \longrightarrow & A \end{array}$$

where $I \subset A$ is an ideal of square zero, the dotted arrow exists and makes the diagram commute. In other words, every map $S \to A/I$ lifts to a map $S \to A$. With this in hand we can state a final characterisation of smoothness.

Proposition 10.11. *For a ring map $R \to S$, being smooth is equivalent to being of finite presentation and formally smooth.*