PICARD GROUP, CARTIER DIVISORS AND TOPICS

The main goal of this talk is to introduce Picard group, Weil divisors and Cartier divisors and proof that they are actually equal for some schemes

§1 PICARD GROUP

Let $(X, \mathcal{O}_X)$ be a scheme.

$$\mathrm{Pic}(X) := \left\{ \begin{array}{l} \text{isomorphism classes of} \\ \text{invertible sheaves on } X \end{array} \right\} \text{ with group operation } \otimes$$

First we will recall what are invertible sheaves. There are two equivalent definitions:

"Categorical" definition: Isomorphism classes of $\mathcal{O}_X$-modules form a monoid with respect to tensor product. The identity element is $\mathcal{O}_X$. Invertible sheaf is an invertible element in this monoid.

$\Rightarrow$ L is invertible if $\exists M \in \mathcal{O}_X\text{-Mod}$ s.t. $L \otimes_{\mathcal{O}_X} M = \mathcal{O}_X$

or equivalently: $L \otimes_{\mathcal{O}_X} L^\vee \longrightarrow \mathcal{O}_X$, where $L^\vee : \mathrm{Hom}(L, \mathcal{O}_X)$ is an iso

"Geometric" definition: Invertible sheaves are locally free $\mathcal{O}_X$-modules of rank 1.

This resembles the definition of vector bundles of rank 1 on manifolds/topological spaces.
Therefore they are sometimes called line bundles

For sheaves of finite pres. these definitions are equivalent:
"$\Rightarrow$" Suppose $\mathcal{L} \otimes_{\mathcal{O}_X} M \cong \mathcal{O}_X$ for some M. We can look at stalks

$$\mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} M_x \cong \mathcal{O}_{X,x}$$

Since $\mathcal{O}_{X,x}$ is local ring and $\mathcal{L}_x$ is of finite presentation we can use corrolary of Nakayama which says that in this case $\mathcal{L}_x \cong \mathcal{O}_{X,x}$. Therefore $\mathcal{L}$ is locally free of rank 1.

"<=" Suppose $\mathcal{L}$ is locally free of rank 1. Then the dual $\mathcal{L}^\vee$ is also locally free of rank 1.

If $\mathcal{L}|_U \cong \mathcal{O}_U$ then $\mathcal{L}^\vee_U := \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, \mathcal{O}_U) \cong \mathcal{O}_U$.

$\Rightarrow \mathcal{L} \otimes \mathcal{L}^\vee|_U \cong \mathcal{O}_U \otimes \mathcal{O}_U \cong \mathcal{O}_U$

Therefore $\mathcal{L}^\vee$ is the inverse of $\mathcal{L}$.

Note: For the second part we do not need the finite pres. But if $\mathcal{L}$ is not of fin. presentation we can teorehically find sheaves invertible in the monoid $(\mathcal{O}_X\text{-Mod}, \otimes)$ which is not locally free of rank 1.

Examples of Picard groups:

- $\mathrm{Pic}(\mathbb{P}^n(k)) \cong \mathbb{Z}$: invertible sheaves for $\mathbb{P}^n(k)$ are the twisting sheaves $\mathcal{O}(m)$ with group operation
$$\mathcal{O}(n) \otimes_{\mathcal{O}_X} \mathcal{O}(m) \cong \mathcal{O}(n+m)$$
- $\mathrm{Pic}(\mathrm{Spec}\, \mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers for algebraic number field K, is the ideal class group
J_K / P_K = fractional ideals / principal fractional ideals
→ for $K = \mathbb{Q}(\sqrt{5})$: $\mathrm{Pic}(\mathrm{Spec}\, \mathcal{O}_K) \cong \mathbb{Z}/2\mathbb{Z}$

§ DIVISORS

The idea of divisors comes from complex analysis.
A divisor on a compact Rieman surface X is a finite linear combination of points (codimension 1 submanifolds) with integer coeficients.

We can also define divisor of meromorphic funchion

→ **Meromorphic funchion** : holomorphic funchion except for a set of isolated points. Intuitively it is a ratio of two holomorphic funchions f/g.

→ We can looke at zeros and poles of meromorphic funchions. Zeros are just **zeros** of f and **poles** are zeros of g.

THM: If f is a meromorphic funchion on nbhd of z_0 then there exists an integer n s.t. $(z-z_0)^n f(z)$ is holomorphic and nonzero in a nbhd of z_0. If $n>0$ then z_0 is a **pole of order** n. If $n<0$ then z_0 is a **zero of order** $|n|$

→ The **divisor** of nonzero meromorphic funchion f is def. as

$$(f) := \sum_{p \in X} \mathrm{ord}_p(f)\, p$$

where $\mathrm{ord}_p(f)$ for zeros is their degree, for poles minus degree

→ Divisors of the form (f) are called **principal divisors**

§2 WEIL DIVISORS

All schemes are considered Noetherian.
Weil divisors comes from the idea of divisor as linear comb. of codimension 1 closed subspaces.

Note: $\overline{\{x\}}$ is of codimension 1 if and only if $\dim \mathcal{O}_{x,X} = 1$. We will then say that x is **point of codimension 1.**

Definition 2.1. : Let X be a Noetherian integral scheme. **Weil divisor** is a formal finite sum

$$Z = \sum_{x \in X} n_x [x]$$

where $n_x \in \mathbb{N}$ and x is point of codimension 1.

→ Weil divisors form a group with pointwise addition denoted Z^1

For general Noetherian schemes we can define **cycle**

$$Z = \sum_{x \in X} n_x [x]$$

It is a formal finite sum of irreducible closed subset. Each such subset can be expressed by $\overline{\{x\}}$ for some x.
The union of the $\overline{\{x\}}$ such that $n_x \neq 0$ is called **support** of Z and is denoted by **Supp Z**.
We say that a cycle Z is of **codimension 1** if the irreducible components of Supp Z are of codimension 1 in X.
The cycles of codimension 1 form a subgroup $Z^1(X)$ of cycles on X.

→ Možná přidat Lemma 2.5.

Example / Definition 2.3.

Let X be normal Noetherian scheme, with generic point η and $K(X) = \mathcal{O}_{X,\eta}$ - the field of rational functions on X.
Then for a point $x \in X$ of codimension 1 the $\mathcal{O}_{X,x}$ is DVR
- local Noetherian ring, normal and of dimension 1

We have $\mathrm{Frac}(\mathcal{O}_{X,x}) = K(X)$. Therefore there is a normalized valuation map: $\mathrm{mult}_x : K(X)^* \to \mathbb{Z}$ associated to $\mathcal{O}_{X,x}$
Let $f \in K(X)$ be a nonzero rational function, then we set

$$(f) : \sum_{x \in X, \dim \mathcal{O}_{X,x} = 1} \mathrm{mult}_x(f)[x]$$

This is Weil divisor (we can prove that this sum must be finite). Such divisor is called **principal divisor**.
As mult_x is additive, we have $(fg) = (f) + (g)$, they form subgroup of Z^1.

Definition 2.4.

Let X be a normal Noetherian scheme. We denote the quotient of $Z^1(X)$ by the subgroup of principal divisors by $Cl(X)$

Example 2.5.

Let p be a prime and consider

$$X = \operatorname{Spec}\mathbb{Z}[x,y]/(xy-p)$$

This is flat, integral, 2-dimensional scheme over $\operatorname{Spec}\mathbb{Z}$.

Since $\dim X = 2$, Weil divisors are linear combination of codimension-1 subschemes, i.e. curves.

Every curve in X is either the closure of a closed point in the generic fiber or a irreducible component of some special fiber

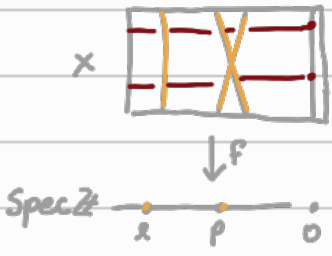
→ Generic fibre (fiber over generic point $(0) \subseteq \mathbb{Z}$)

$$X_{\mathbb{Q}} := X \otimes_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{Q} \cong \operatorname{Spec}\mathbb{Q}[x,y]/(xy-p) \cong \operatorname{Spec}\mathbb{Q}[x,x^{-1}] \cong \mathbb{G}_{m,\mathbb{Q}}$$

→ Special fiber (fiber over closed point $(p) \subseteq \mathbb{Z}$)

- $\ell \neq p$: $X_{\mathbb{F}_\ell} := X \otimes_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{F}_\ell \cong \operatorname{Spec}\mathbb{F}_\ell[x,y]/(xy-p) \cong \mathbb{G}_{m,\mathbb{F}_\ell}$
- p: $X_{\mathbb{F}_p} := X \otimes_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{F}_p \cong \operatorname{Spec}\mathbb{F}_p[x,y]/(xy-p) \cong \operatorname{Spec}\mathbb{F}_p[x,y]/(xy)$
  → It is reducible, it has two irreducible components → cross

Idea: These are "horizontal" and "vertical" curves



$\dim f(\cdot) = 0$

$\dim f(\cdot) = 1$

§3 CARTIER DIVISORS

The idea behind Cartier divisors comes from the other point of view on divisors - using meromorphic functions.

First we need to define what should be meromorphic function for any scheme

Note: A ring, $f \in A$ is called regular element if it's not a zero divisor.

→ Being regular is equivalent of being regular in every localization → Being regular is local property

We denote $R(A)$ the (multiplicative) group of regular elements.

We denote $\operatorname{Frac}(A) := R(A)^{-1}A$

Definition 3.1.

Let $(X, \mathcal{O}_X)$ be a scheme. We can define a sheaf

$$\mathcal{R}_X(U) := \{a \in \mathcal{O}_X(U) \mid a_x \in R(\mathcal{O}_{X,x}), \forall x \in U\}$$

→ Clearly $\mathcal{R}(U) \subseteq R(\mathcal{O}_X(U))$. For U affine equality holds

Definition 3.2.

We define presheaf of algebras $\mathcal{K}'_X$ containing $\mathcal{O}_X$ by following property: for any open subset U of X we have $\mathcal{K}'_X(U) = \mathcal{R}_X(U)^{-1}\mathcal{O}_X(U)$. (In particular for U affine $\mathcal{K}'_X(U) = \mathrm{Frac}(\mathcal{O}_X(U))$)

Definition 3.3.

$\mathcal{K}_X$ is the sheafification of $\mathcal{K}'_X$ called **sheaf of stalks of meromorphic functions**. Element of $\mathcal{K}_X(X)$ is called **meromorphic function**. We denote the subsheaf of invertible elements of $\mathcal{K}_X$ by $\mathcal{K}_X^*$.

In complex analysis two non-zero meromorphic functions have the same divisor if they differ by a multiple of invertible holomorphic function. This idea motivates the definition of Cartier divisors

Definition 3.4.

Let X be a scheme. We denote the group $H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \cong \mathcal{K}_X^*/\mathcal{O}_X^*(X)$ by $\mathrm{Div}(X)$, elements of $\mathrm{Div}(X)$ are called **Cartier divisors** on X.

Let $f \in H^0(X, \mathcal{K}_X^*) \cong \mathcal{K}_X^*(X)$, its image in $\mathrm{Div}(X)$ is called **principal Cartier divisor** and denoted by $\mathrm{div}(f)$.

The group law on $\mathrm{Div}(X)$ is noted additively. We say that two Cartier divisors are **linearly equivalent** if $D_1 - D_2$ is principal.

Cartier divisor D is called **effective** if it is in the image of the canonical map map $H^0(X, \mathcal{O}_X \cap \mathcal{K}_X^*) \to H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$. We then write $D \geq 0$ or $D \in \mathrm{Div}_+(X)$.

By definition, we can present a Cartier Divisor D by a system $\{(U_i, f_i)\}$, where the U_i are open subsets of X forming a covering of X, f_i is the quotient of two regular elements of $\mathcal{O}_X(U_i)$, and f_i & f_j differ by multiplicative factor in $\mathcal{O}_X(U_i \cap U_j)^*$, i.e. $f_{i|_{U_i \cap U_j}} \in f_{j|_{U_i \cap U_j}} \mathcal{O}_X(U_i \cap U_j)^*$, $\forall i, j$.

Similarly, two systems $\{(U_i, f_i)_i\}$, $\{(V_j, g_j)_j\}$ represent the same Cartier divisor if f_i and g_j differ by multiplicative factor in $\mathcal{O}_X(U_i \cap V_j)^*$.

We denote by $\mathrm{CaCl}(X)$ the group of isomorphism classes of Cartier divisors modulo the linear equivalence relation.

The sum of two Cartier divisors $D_1 + D_2$ is represented by $\{(U_i \cap V_j, f_i g_j)_{i,j}\}$.
Principal divisor can be represented by $\{(X, f)\}$
Therefore principal divisors are again the ones coming from functions

Effective Cartier divisor D can be represented by $\{(U_i, f_i)_i\}$, $f_i \in \mathcal{O}_X(U_i)$.
We can define an ideal sheaf $I_D \subseteq \mathcal{O}_X$ by

$$I_D|_{U_i} := f_i \mathcal{O}_X|_{U_i}$$

On $U_i \cap U_j$ we have

$$f_i \mathcal{O}_{X|_{U_i \cap U_j}} \cong u_{ij} f_i \mathcal{O}_{X|_{U_i \cap U_j}} \cong f_j \mathcal{O}_{X|_{U_i \cap U_j}} \quad \text{since } u_{ij} \in \mathcal{O}_X(U_i \cap U_j)^*.$$

$\Rightarrow$ local ideals agree and glue to a global sheaf
locaLy defined by one element $\Rightarrow$ it is quasi-coherent
Every quasi-coherent subsheaf of ideals on $\mathcal{O}_X$ corresponds to closed subscheme of $\mathcal{O}_X$.
Consider now the map:

$$\mathcal{O}_X|_{U_i} \xrightarrow{f_i} f_i \mathcal{O}_X|_{U_i}$$

It is a bijection: • surjective by definition
• injective since f_i is non-zerodivisor (regular)
$\Rightarrow$ Locally free of rank 1 $\Rightarrow$ Invertible sheaf of $\mathcal{O}_X$-modules
Therefore we can equivalently define effective Cartier divisors as

Definition 3.5.

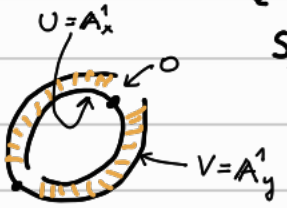
An effective Cartier divisor on scheme X is a closed subscheme $D \subset S$ whose ideal sheaf $I_D \subset \mathcal{O}_X$ is an invertible $\mathcal{O}_X$-module.

It is also locally principal closed subscheme of S - closed subscheme whose sheaf of ideals is locally generated by single element.

Example

We will consider the projective scheme $\mathbb{P}^1_k$.

"We glue two affine line by associating x and y^{-1}"



$$\begin{array}{ccccc} \mathrm{Spec}\, k[t,t^{-1}] & + \hookleftarrow & \times & & \mathrm{Spec}\, k[x] \\ \| & & & & \| \\ t^{-1} & \| & G_{m,k} & \to & \mathbb{A}^1_{x,k} \\ \uparrow & & \downarrow & & \downarrow \\ y & & & \ulcorner & \downarrow \\ \mathrm{Spec}\, k[y] & \cong & \mathbb{A}^1_{y,k} & \to & \mathbb{P}^1_k \end{array}$$

We would like to get 0 as a vanishing locus of global regular function. Unfortunately $H_0(X, \mathbb{P}^1_k) = k$ so this is not possible.

But we can get a Cartier divisor with this property.

It is going to be represente by $\{(U, x), (V, y^{-1})\}$.

§4 How all of them connect

First we look at connection of Cartier divisors and Picard group

Similarly as for effective divisors we can construct for any Cartier divisor a sub-$\mathcal{O}_X$-module $\mathcal{O}_X(D) \subseteq \mathcal{K}_X$ defined by

$$\mathcal{O}_X(D)_{|U_i} = f_i^{-1}\mathcal{O}_X(U_i)$$

Clearly independent of the choice of the representing system and we see that it is invertible sheaf.

In this notation D is effective iff $\mathcal{O}_X(-D) = \mathcal{I}_D \subseteq \mathcal{O}_X$.

By definition the map $p: D \mapsto \mathcal{O}_X(D)$ is additive

$$p(D_1 + D_2) = \mathcal{O}_X(D_1)\mathcal{O}_X(D_2) \cong \mathcal{O}_X(D_1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D_2)$$

In fact following theorem holds

Theorem 4.1.

Let X be reduced Noetherian scheme, then there is an isomorphism

$$\mathrm{CICa}(X) \longrightarrow \mathrm{Pic}(X)$$

Proof:

Homomorphism: We have seen that it is additive and principal Cartier divisors are clearly sent to $\mathcal{O}_X$, since they are repres. by (X, f)

Injectivity: We need to prove that any $D \in \mathrm{Ker}\, p$ is principal.
Suppose $D \in \mathrm{Ker}\, p$. Then there exists $f \in H^0(X, \mathcal{O}_X(D) \subseteq H^0(X, \mathcal{K}_X)$ s.t. $\mathcal{O}_X(D) = f\mathcal{O}_X$. Which implies D is represented by $(X, f) \Rightarrow$ principal.
(This is true for all schemes)

Surjectivity: 1. We prove that for any scheme $\mathrm{im}(\mathrm{CICa}(X)) \subseteq \mathrm{Pic}(X)$ corresponds to invertible sheaves contained in $\mathcal{K}_X$:
By construction $p(D) = \mathcal{O}_X(D) \subseteq \mathcal{K}_X$ is an invertible subsheaf.
Let $\mathcal{L} \subseteq \mathcal{K}_X$ be an invertible subsheaf. Let $\{U_i\}_i$ be an open covering of X such that $\mathcal{L}_{|U_i}$ free and generated by $f_i \in \mathcal{K}_X'(U_i)$, which must be invertible, therefore $f_i \in \mathcal{K}_X^*(U)$. Then the Cartier divisor D associated to $\{(U_i, f_i)\}$ verifies $p(D) = \mathcal{L}$.
2. We prove that in case of X being Noetherian and reduced actually every invertible sheaf is isomorphic to sub-$\mathcal{O}_X$-module of $\mathcal{K}_X$:

Since X is Noetherian it has finitely many irreducible components
$\Rightarrow$ finitely many generic points $\xi_1, \dots, \xi_n$ with finite n.
Each ξ_j an open nbhd U_j contained in $\overline{\{\xi_j\}} \setminus \bigcup_{i \neq j} \overline{\{\xi_i\}}$. We can take U_j small enough that $\mathcal{L}|_{U_j}$ is free. Now take $U = \bigcup_{1 \leq j \leq n} U_j$, U_j's are pairwise disjoint so $\mathcal{L}|_U$ is also free.
Now we use the fact that X is also reduced, thus generic points are associated points. For Noetherian scheme and open subset U containing all associated points we know following (not proven here)
Let $i : U \to X$ be can. injection: $\cdot\ \mathcal{L} \to i_* (\mathcal{L}|_U)$ is an injection
$\cdot\ \mathcal{K}_X \to i_* \mathcal{K}_U$ is an isomorphism

Therefore we have

$$\mathcal{L} \hookrightarrow i_*(\mathcal{L}|_U) \simeq i_* \mathcal{O}_U \hookrightarrow i_* \mathcal{K}_U \simeq \mathcal{K}_X$$

which proves that $\mathcal{L}$ sub-$\mathcal{O}_X$-module of X

Now we want to look at relation of Weil and Cartier divisors

Let X be locally Noetherian scheme, x point of codimension 1.
For $D \in \mathrm{Div}(X)$ represented by f/g at x (as element of $(\mathcal{K}_X^* / \mathcal{O}_X^*)_x$) we can define what is $\mathrm{mult}_x(D)$ as the lenght of $\mathcal{O}_{X,x}$-module $\mathcal{O}_{X,x}/f\mathcal{O}_{X,x}$ minus lenght of $\mathcal{O}_{X,x}/g\mathcal{O}_{X,x}$. Which in this seting will be always finite
This should represent multiplicity of zeros/poles of this D as function.

Thus we can define

$$[D] = \sum_{x \in X, \dim \mathcal{O}_{X,x} = 1} \mathrm{mult}_x(D)\,[x] \in Z^1(X)$$

Which gives a group homomorphism

$$\mathrm{Div}(X) \to Z^1(X), \quad D \mapsto [D]$$

which sends effective divisors to positive cycles.

Proposition 4.2.

Let X be normal Noetherian scheme

- Let $f \in K(X)^* = H^0(X, \mathcal{K}^*)$. Then Weil divisor $(f) \in Z^1$ coincides with the image of principal divisor $\mathrm{div}(f)$.

- The homomorphism $\mathrm{Div}(X) \to Z^1(X)$ is injective and induces an injective homomorphism $\mathrm{CaCl}(X) \to \mathrm{Cl}(X)$

Proposition 4.3.

Let X be a Noetherian regular integral (and hence normal) scheme. Then the homomorphisms

$$\mathrm{Div}(X) \to Z^1(X), \quad \mathrm{CaCl}(X) \to \mathrm{Cl}(X)$$

are isomorphisms.

Proof idea: It is enough to prove surjectivity of $\mathrm{Div}(X) \to Z^1(X)$. We know that irreducible closed subsets of dim 1 generate $Z^1(X)$. For such subset E we can construct a preimage D. This preimage will satisfy $\mathrm{mult}_\xi(D) = 1$ for ξ generic point of E and 0 elsewhere.

§5 CONCLUSION

