Models can be normies

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Abstract

We show the construction of a regular model for a smooth, projective, connected curve C over the field of fractions K of an affine Dedekind scheme $S = \mathrm{Spec}(A)$.

Definition 1. *Let S be a Dedekind scheme. We call an integral, projective, flat S-scheme $\pi : X \to S$ of $\dim X = 2$ a **fibered surface over** S. We call X_η the generic fiber, and X_s (with $s \in S$ closed) a closed fiber. We will say that X is a **normal (regular) fibered surface** if X is normal (regular).*

Definition 2. *We will call a regular fibered surface $X \to S$ over a Dedekind scheme S of dimensional 1 an **arithmetic surface**.*

Definition 3. *Let S be a Dedekind scheme of dimenisonal 1, with function field K. Let C be a normal, connected, projective curve over K. We call a normal fibered surface $\mathcal{C} \to S$ together with an isomorphism $f : \mathcal{C}_\eta \xrightarrow{\sim} C$ a **model** of C over S. We will say that $\mathcal{C}$ is a **regular model** of C if $\mathcal{C}$ is regular.*

Example. Blowing up a closed point on the special fiber of $\mathbb{P}^1_{\mathbb{Z}}$ produces a regular, non-smooth model of $\mathbb{P}^1_{\mathbb{Q}}$.

Let $S = \mathrm{Spec}(\mathbb{Z})$, $C = \mathbb{P}^1_{\mathbb{Q}}$. The standard model is:

$$\mathcal{X}_0 = \mathbb{P}^1_{\mathbb{Z}} = \mathrm{Proj}(\mathbb{Z}[T_0, T_1]) \longrightarrow \mathrm{Spec}(\mathbb{Z}),$$

which is smooth over $\mathrm{Spec}(\mathbb{Z})$.

Fix a prime $p \in \mathbb{Z}$ and consider the closed point $P \in \mathbb{P}^1_{\mathbb{Z}}$ on the special fiber $(\mathbb{P}^1_{\mathbb{Z}})_p = \mathbb{P}^1_{\mathbb{F}_p}$ given by $[T_0 : T_1] = [1 : 0]$. Let $t = \frac{T_1}{T_0}$. On the affine chart $U_0 = D(T_0)$, P corresponds to the maximal ideal $\mathfrak{m} = (t, p) \subset \mathbb{Z}[t]$.

Define $\mathcal{X} := \mathrm{Bl}_P(\mathbb{P}^1_{\mathbb{Z}})$ to be the blow up of $\mathbb{P}^1_{\mathbb{Z}}$ at P. $\mathcal{X}$ is covered by $U_1 = D(T_1)$, $\mathcal{V}_1 = \mathrm{Spec}\,\mathbb{Z}[t]\left[\frac{p}{t}\right] \cong \mathrm{Spec}\,\mathbb{Z}[t, u]/(tu - p)$ and $\mathcal{V}_2 = \mathrm{Spec}\,\mathbb{Z}[t]\left[\frac{t}{p}\right] \cong \mathrm{Spec}\,\mathbb{Z}[v]$. It's not hard to see $\mathcal{X}$ is a regular model of $\mathbb{P}^1_{\mathbb{Z}}$. Consider its special fiber $\mathcal{X}_p = \mathcal{X} \times_{\mathbb{Z}} \mathbb{F}_p$. On the chart $\mathcal{V}_1$, the special fiber is

$$(\mathcal{V}_1)_p = \mathrm{Spec}(\mathbb{F}_p[t, u]/(tu)),$$

which shows $\mathcal{X}_p$ is not smooth over $\mathbb{F}_p$.

Lemma 1 (Liu 8.2.37)**.** *Let $\mathcal{X}$ be a locally Noetherian scheme. Suppose that for every integral closed subscheme Y of $\mathcal{X}$, the set $\mathrm{Reg}(Y)$ contains a non-empty open subset. Then $\mathrm{Reg}(\mathcal{X})$ is open.*

Proposition 1. *For $X \to \operatorname{Spec} \mathcal{O}_K$ a fibered surface with smooth generic fiber X_η, $\mathrm{Reg}(x) := \{x \in X \mid X \text{ is regular at } x\}$ is open.*

Proof. The smooth locus $\mathrm{sm}(X) := \{x \in X \mid X \text{ is smooth at } x\}$ is open in X (Liu 6.2.12). So $\pi(X \setminus \mathrm{sm}(X))$ is a closed subset of $\operatorname{Spec} \mathcal{O}_K$, which means a finite number of closed points, let's say $\{s_1, \ldots, s_n\}$. So $\mathrm{sm}(X)$ contains X_η and all X_s except $s \in \{s_1, \ldots, s_n\}$.

We then reduce to the local case and use the above lemma to finish the proof. For $x \in X_s$, define $A := \mathcal{O}_{K,s}$, $X_A := X \times_{\operatorname{Spec} \mathcal{O}_K} \operatorname{Spec} A$. As a topology space, X_A carries the subspace topology from X, with underlying set $|X_A| = |X_\eta| \cup |X_s|$. Moreover, for every $z \in X_A$, $\mathcal{O}_{X_A,z} \cong \mathcal{O}_{X,z}$ as it's just a further localization, so the regularity is preserved. If we prove that $\mathrm{Reg}(X_A)$ is open in X_A, then for $x \in X_s \cap \mathrm{Reg}(X)$, there is an open neighborhood $U \subseteq X$ of x s.t. $U \cap X_A \subseteq \mathrm{Reg}(X_A)$. Then $U' = U \setminus \bigcup_{t \in \{s_1,\ldots,s_n\} \setminus \{s\}} X_t$ is open in X, and $U' \subseteq \mathrm{Reg}(X)$.

By the lemma, we take an arbitrary integral closed subscheme $Y \subseteq X_A \to \operatorname{Spec} \mathcal{O}_{K,s}$, then $\mathrm{Im}(Y \to \operatorname{Spec} \mathcal{O}_{K,s})$ is closed in $\operatorname{Spec} \mathcal{O}_{K,s}$. If $H = \{s\}$, then $Y = Y_s$. And $\mathrm{Reg}(Y_s) = \mathrm{sm}(Y_s)$ as $k(s)$ is perfect, so $\mathrm{Reg}(Y)$ is itself open. If $\eta \in H$, Y_η is open in Y, with $\mathrm{Reg}(Y_\eta) = Y_\eta \cap \mathrm{Reg}(Y)$. So $\mathrm{Reg}(Y_\eta)$ is open in $\mathrm{Reg}(Y)$. □

Proposition 2. *For a smooth, projective, connected curve C over the field of fractions K of an affine Dedekind scheme $S = \operatorname{Spec} A$, a regular model exists.*

Proof. We do it in 4 steps:

$\mathcal{C}_0$ naive model

↓ taking the scheme-theoretic closure

$\mathcal{C}_1$ fibered surface

↓ normalization

$\mathcal{C}_2$ model

↓ desingularization

$\mathcal{C}_3$ regular model

Step 1: Assume $C = \mathrm{Proj}(K[T_0, \ldots, T_N]/I)$ for $I = (F_1, \ldots, F_m)$ the homogeneous ideal. Find $a_i \neq 0 \in A$ st. $\widetilde{F}_i = a_i F_i \in A[T_0, \ldots, T_N]$. Then we define $\mathcal{C}_0$ as

$$\mathcal{C}_0 := \mathrm{Proj}\left(\frac{A[T_0, \ldots, T_N]}{(\widetilde{F}_1, \ldots, \widetilde{F}_m)}\right) \to \operatorname{Spec} A$$

with $(\mathcal{C}_0)_\eta \cong C$, i.e. $C = \mathcal{C}_0 \times_A K$.

Step 2: Let $\mathcal{C}_1$ be the scheme-theoretic closure of $C \to \mathcal{C}_0$, with reduced structure. Recall that it's defined via the sheaf of ideals $\mathcal{I} = \ker(\mathcal{O}_{\mathcal{C}_0} \to f_*\mathcal{O}_C)$. Locally on $D(T_i) = \operatorname{Spec} B_i$,

$$I_i = \ker(B_i \to B_i \otimes_A K) = \{b \in B_i \mid ab = 0 \text{ for some } 0 \neq a \in A\}$$

Thus $\mathcal{C}_1 \cap D(T_i) = \operatorname{Spec}(B_i/I_i)$, that is making the coordinate ring A-torsion free. Then we take the reduced structure, i.e., $\mathcal{C}_1 \cap D(T_i) = \operatorname{Spec}((B_i/I_i)/\sqrt{0}) = \operatorname{Spec}(B_i/\sqrt{I_i})$.

Since C is irreducible, taking its closure $\mathcal{C}_1$ is also irreducible, thus $\mathcal{C}_1$ is integral. Flatness comes from A-torsion free. Since $\mathcal{C}_1 \subseteq \mathcal{C}_0$ a closed subscheme, $\mathcal{C}_1$ is projective. Let $K(C)$ be the function field of C, then $\dim \mathcal{C}_1 = \dim \operatorname{Spec} A + \operatorname{tr.deg}_K K(C) = 1 + 1 = 2$ (Sta, Tag 02JX). So $\mathcal{C}_1$ is a fibered surface over S.

Step 3: We want to make it a model by the process "normalization".

Definition 4 (Sta, Tag 0BAK)**.** *Let $f : Y \to X$ be a qcqs. The normalization of X in Y is the morphism $\nu : \widetilde{X} \to X$ defined by $\widetilde{X} = \underline{\operatorname{Spec}}_X(\mathcal{A})$, where $\mathcal{A} \subseteq f_*\mathcal{O}_Y$ is the quasi-coherent sheaf of $\mathcal{O}_X$-algebras given by the integral closure of $\mathcal{O}_X$ in $f_*\mathcal{O}_Y$.*

Remark. In affine case, it is simply taking the integral closure. Over every affine open $U = \operatorname{Spec}(R) \subseteq X$, $\nu^{-1}(U)$ is literally $\operatorname{Spec}(\mathcal{A}(U))$.

For $C \to \mathcal{C}_1$, $\mathcal{C}_1$ is covered by affine charts $U_i = \operatorname{Spec}(B_i)$, and we define $\mathcal{A}$ via taking the integral closure of B_i in $B_i \otimes_A K = \operatorname{Quot}(B_i)$. Since B_i integral and A-torsion free, $A \subseteq B_i$ and $B_i \otimes_A K \subseteq \operatorname{Quot}(B_i)$. Since $U_i \cap C = \operatorname{Spec} B_i \otimes_A K$ and C is normal, $B_i \otimes_A K$ is integrally closed in $\operatorname{Quot}(B_i)$, which implies the integral closure of B_i in $B_i \otimes_A K$ is just $\widetilde{B}_i$, the integral closure of B_i in $\operatorname{Quot}(B_i)$, i.e. $\nu^{-1}(\operatorname{Spec} B_i) = \operatorname{Spec} \widetilde{B}_i$ for $\nu : \mathcal{C}_2 \to \mathcal{C}_1$, the normalization of $\mathcal{C}_1$ in C.

By construction, $\mathcal{C}_2$ is integral and normal. It's not hard to see $\nu : \mathcal{C}_2 \to \mathcal{C}_1$ is a finite morphism (Sta, Tag 0AVK). A key point is that for $f : C \to \mathcal{C}_1$, $k(\eta)/k(f(\eta)) = K(C)/K(\mathcal{C}_1) = 0$, which means this field extension is finitely generated (in the trivial sense). Thus $\mathcal{C}_2 \to \operatorname{Spec} A$ is projective. Using the same approach as in step 2, we know $\dim \mathcal{C}_2 = 2$.

Step 4: We use the theory of desingularization to make it a regular model.

Definition 5 (Liu, 8.3.39)**.** *Let X be a reduced locally Noetherian scheme. A desingularization in the strong sense of X is a proper birational morphism $\pi : \widetilde{X} \to X$ with $\widetilde{X}$ a regular scheme and π restricts to an isomorphism over the regular locus: $\pi^{-1}(\operatorname{Reg}(X)) \xrightarrow{\sim} \operatorname{Reg}(X)$.*

We use Lipman's construction: Let us consider the following sequence of proper birational morphisms:

$$\cdots \to X_{n+1} \to X_n \to \cdots \to X_3 \to X_2 \to X_1 \to X$$

where $X_1 \to X$ is the normalization of X, and for every $i \geq 1$, $X_{i+1} \to X_i$ is the composition of the blowing-up $X_i' \to X_i$ of the singular locus $X_i \backslash \operatorname{Reg}(X_i)$ (which

is closed) endowed with the reduced structure, and of the normalization $X_{i+1} \to X_i'$. Note that each X_i' is still a model as blowing up preserves integrality. The sequence stops at n when X_n is regular.

Taking $X_1 \to X$ as $\mathcal{C}_2 \to \mathcal{C}_1$, this process terminates in finite steps and gives a regular model of C. □

Example. Let $A = \mathbb{Z}_{(3)}$ with fraction field $K = \mathbb{Q}$ and residue field $\kappa(3) = \mathbb{F}_3$. Consider the smooth projective elliptic curve C over $\mathbb{Q}$ given by:

$$C : Y^2Z = X^3 + 9Z^3 \quad \text{in } \mathbb{P}^2_{\mathbb{Q}}$$

$$\mathcal{C}_0 = \mathcal{C}_1 = \operatorname{Proj}\left(\frac{\mathbb{Z}_{(3)}[X,Y,Z]}{Y^2Z - X^3 - 9Z^3}\right) \longrightarrow \operatorname{Spec}(\mathbb{Z}_{(3)})$$

(since $\mathcal{C}_0$ is already integral and A-torsion-free). It has generic fiber C and special fiber $(\mathcal{C}_1)_3 : Y^2Z = X^3$ in $\mathbb{P}^2_{\mathbb{F}_3}$. One can check $\mathcal{C}_1$ is already normal but does not regular at the point $P = [0:0:1]$. To get a regular model, we blow up at the singular locus $\mathcal{C}_2 \setminus \operatorname{Reg}(\mathcal{C}_2) = \{P\}$ (defined by the ideal $\mathfrak{m}_P = (x, y, 3)$). And $\mathcal{C}_3 = \operatorname{Bl}_P(\mathcal{C}_1) \longrightarrow \operatorname{Spec}(\mathbb{Z}_{(3)})$ gives a regular model of C.

References

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