Implications of Breuil-Herzig-Hu-Morra-Schraen's conjectures on Zábrádi's functor

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Abstract. — Let ρ be an n-dimensional representation of $\mathcal{G}_{\mathbb{Q}_p}$ over $\overline{\mathbb{F}_p}$. When ρ is generic and a good conjugate, the article [Bre+21] introduces the notion of compatibility with ρ for an admissible representation of $\mathrm{GL}_n(\mathbb{Q}_p)$. In *loc. cit.*, the five authors also question whether one could recover a representation of $\mathcal{G}_{\mathbb{Q}_p}^{n-1}$, called $\overline{L}^{\boxtimes}(\rho)$ and constructed from ρ, from some Π compatible with ρ by using Zábrádi's functor $\mathbf{V}_\Delta$ (see [JZ25]). We give a range of results, for an arbitrary Π verifying some "weak" compatibilities with ρ, about how badly $\mathbf{V}_\Delta(\Pi)$ behaves. In particular, when ρ is reducible and $n \geq 3$, no Π compatible with P_ρ can verify $\mathbf{V}_\Delta(\Pi) \simeq \overline{L}^{\boxtimes}(\rho)$.

1   Introduction / recollection

While the mod p local Langlands correspondence is well understood for $\mathrm{GL}_2(\mathbb{Q}_p)$ (see [Bre03a] [Col10], etc), the situation for $\mathrm{GL}_n(\mathbb{Q}_p)$ remains at the step of groping into specific examples, especially the ones coming from the global picture. One of the main issues is that the theory of smooth admissible finite length $\mathrm{GL}_n(\mathbb{Q}_p)$-representations modulo p lacks both a classification of irreducibles as given by [BL94] [Bre03a] [Bre03b] for $\mathrm{GL}_2(\mathbb{Q}_p)$ and the study of extensions as in [Col10, §VII]. Partial answers can be found both in [Her11] for the classification and in [Hau16] for the extensions' study, but they don't tackle supercuspidal representations.

Let ρ be an n-dimensional representations ρ of $\mathcal{G}_{\mathbb{Q}_p}$ over an algebraic extension $\mathbb{F}$ of $\mathbb{F}_p$. Recently, five authors have formulated in [Bre+21] some conjectures about the shape of representations associated by the global picture to ρ, when the latter is generic and a good conjugate. More precisely, they define in [Bre+21, Def. 2.4.2.7] the notion of a smooth admissible representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ over $\mathbb{F}$ compatible with ρ. The number of Jordan-Hölder components of such Π are prescribed by the minimal standard parabolic P_ρ whose $\mathbb{F}$-points contain the image of ρ; this P_ρ also determines how these components fit in the classification of [Her11] (see [Bre+21, Def. 2.4.1.5]). The combinatoric of Π's subquotients is dicted by the smallest algebraic subgroup $\widetilde{P}_\rho$ containing the Levi M_{P_ρ} and whose $\mathbb{F}$-points contain the image of ρ. Finally, compatibility imposes the image of the irreducible components by Breuil's functor $\mathbf{V}_\xi$ among subquotients of $\overline{L}^{\otimes}(\rho) := \otimes_{1 \leq i < n} \Lambda^i \rho$.

Here, Breuil's functor $\mathbf{V}_\xi$ is a version of Colmez's functor $\mathbf{V}$ (see [Col10, §IV.3]) defined in [Bre15] that produces ind-finite-dimensional Galois representations from smooth representations of $\mathrm{GL}_n(\mathbb{Q}_p)$. We won't need its precise definition. One current issue of Breuil's functor is that we do not know wether it does produce finite dimensional, nor non zero objects, except for principal series or for $n = 2$.

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Another generalisation of $\mathbf{V}$ has been developed by G. Zábrádi in [Záb18a] for a general reductive group G. This functor $\mathbf{D}_\Delta^\vee$ will be the other actor of this note. Let us recall the philosophy of Zábrádi's functor $\mathbf{D}_\Delta^\vee$. In our situation, Δ is the canonical set of simple roots in the canonical root system of GL_n, verifying $\#\Delta = n - 1$. We identify $[\![1, n-1]\!]$ to Δ by sending i to the character of the torus $\mathrm{Diag}(x_1, \ldots, x_n) \mapsto x_i x_{i+1}^{-1}$ and equip Δ with the associated order. Combining [Záb18a] and [JZ25, Th. A], for any smooth representation Π of $\mathrm{GL}_n(\mathbb{Q}_p)$ over $\mathbb{F}$, of finite length, $\mathbf{D}_\Delta^\vee$ produces a finite projective topological étale $T_{+,\Delta}$-module over $E_\Delta := \cup_{\mathbb{F}_q \subset \mathbb{F}} \mathbb{F}_q[\![X_\alpha \mid \alpha \in \Delta]\!][X_\Delta^{-1}]$. Here $T_{+,\Delta}$ is the monoid of diagonal matrices with decreasing valuations. If Π has a central character, then $\mathbb{Q}_p^\times \mathrm{Id}_n \subset T_{+,\Delta}$ also acts via a character on $\mathbf{D}_\Delta^\vee$. Hence, the slightest generalisation of [Záb18b], produces an equivalence $\mathbb{V}_\Delta$ from $\mathscr{M}\mathrm{od}^{\text{ét}}(T_{+,\Delta}, E_\Delta)$ with "central character" and smooth finite dimensional representations of $(\mathcal{G}_{\mathbb{Q}_p, \Delta} \times \mathbb{Q}_p^\times)$ over $\mathbb{F}$ with "central character", i.e. $\mathbb{Q}_p^\times$ acting via homotethies. Post-composing $\mathbf{D}_\Delta^\vee$ with this equivalence, forgetting the central characters and applying a suitable duality functor provides a functor $\mathbf{V}_\Delta$ to finite dimensional representations of $\mathcal{G}_{\mathbb{Q}_p}^{n-1}$. Two important features are to highlight. First, it seems to keep more information than Breuil's functor by staying with multivariable (φ, Γ)-modules; indeed, [Záb18a] produces a $\mathrm{GL}_n(\mathbb{Q}_p)$-equivariant sheaf on the flag variety $\mathrm{GL}_n(\mathbb{Q}_p)/B(\mathbb{Q}_p)$ from this multivariable (φ, Γ)-module associated to a $\mathrm{GL}_n(\mathbb{Q}_p)$-representation, similiarily to what $D \boxtimes \mathbb{P}^1(\mathbb{Q}_p)$ was for $\mathrm{GL}_2(\mathbb{Q}_p)$ (see [Col10, §II.1]). Second, from this equivariant sheaf, [JZ25] deduces finiteness results that are lacking for Breuil's functor. They don't recover more non-vanishing results than what was known for Breuil's functor.

As stated in [Bre+21, Rem. 1.2.5], the existence of Zábrádi's functor allows an improved conjecture about what the Π associated using global methods looks like. The best scenario would be that Zábrádi's functor produces $\overline{L}^\boxtimes(\rho) := (\boxtimes_{1 \leq i < n}(\Lambda^i \rho))$ and that its restriction to the diagonal embedding of $\mathcal{G}_{\mathbb{Q}_p}$ identifies with $\mathbf{V}_\xi(\Pi)$, which is isomorphic to $\overline{L}^\otimes(\rho)$. Such conjecture fills the following small insatisfaction: the representation $\overline{L}^\otimes(\rho)$ doesn't determine ρ, already in dimension 3, where $\overline{L}^\boxtimes(\rho)$ does.

The first aim of this note is to prove that this scenario never happens in the reducible case, even when downgrading compatibility with ρ to compatibility with $\widetilde{P}_\rho$ (see Theorem 3.5). The rough idea is that Π being compatible with $\widetilde{P}_\rho$ imposes a shortage of Jordan-Hölder components of $\mathbf{V}_\Delta(\Pi)$, in comparison to their profusion in $\overline{L}^\boxtimes(\rho)$. We use the same method to treat a number of irreducible cases (see Theorem 3.6).

One can argue that the condition [Bre+21, Def. 2.4.2.7 (i)] imposes heavy conditions on the image of supersingular by $\mathbf{V}_\Delta$. What would happen if we replaced the *loc. cit.* conditions (i)-(ii) by the single $\mathbf{V}_\xi(\Pi) = \overline{L}^\otimes(\rho)$? This note also gives bad behavior of $\dim \mathbf{V}_\Delta(\Pi)$ under this hypothesis for a toy example; we tackle the case of $\mathrm{GL}_3(\mathbb{Q}_p)$ and ρ being of dimension 3 and Loewy length 3 over $\overline{\mathbb{F}_p}$. For the precise result, see Proposition 5.5.

Notations

In this note, the field $\mathbb{F}$ is an algebraic extension of $\mathbb{F}_p$. It will be equipped with the discrete topology, implying that continuous representations of locally profinite groups are smooth, i.e. have open stabilisers. All inductions should be smooth inductions.

For a finite length representation V of a group G, we denote the socle (resp. cosocle) filtration by $\mathrm{soc}_G^i V$ (resp. $\mathrm{cosoc}_G^i V$) and its graded pieces by $\mathrm{soc}_{i,G} V$ (resp. $\mathrm{cosoc}_{i,G} V$). The Loewy length is the minimum integer i such that $\mathrm{soc}_G^i = V$.

Recall that the Artin map is defined by

$$\mathrm{Art}^{-1} : \mathcal{G}_{\mathbb{Q}_p}^{\mathrm{ab}} \to \widehat{\mathbb{Q}_p^\times}, \ \sigma \mapsto x_\sigma p^{n_\sigma}$$

where $x_\sigma \in \mathbb{Z}_p^\times$ is characterised by $\forall \zeta \in \mu_{p^\infty}, \ \sigma(\zeta) = \zeta^x$. It verifies $\mathrm{Art}^{-1}(\mathrm{W}_{\mathbb{Q}_p}^{\mathrm{ab}}) = \mathbb{Q}_p^\times$, where $\mathrm{W}_{\mathbb{Q}_p}$ is the Weil group.

We denote by ε the smooth character of $\mathbb{Q}_p^\times$ defined by

$$\varepsilon : \mathbb{Q}_p^\times \to \mathbb{F}_p^\times, \ xp^n \mapsto (x \mod p)$$

for $x \in \mathbb{Z}_p^\times$. It extends to $\widehat{\mathbb{Q}_p^\times}$.

We denote by ω the smooth character of $\mathcal{G}_{\mathbb{Q}_p}$ defined as the composition

$$\omega \; : \; \mathcal{G}_{\mathbb{Q}_p} \to \mathcal{G}_{\mathbb{Q}_p}^{\mathrm{ab}} \xrightarrow{\mathrm{Art}^{-1}} \widehat{\mathbb{Q}_p^\times} \xrightarrow{\varepsilon} \mathbb{F}_p^\times.$$

For $G = \mathrm{GL}_n$, we keep the notation $\overline{L}^{\otimes}$ introduced in [Bre+21, §2.1] for the antisymmetric powers of the standard representations over $\mathbb{F}$.

For a profinite group G and a discrete field k, let $\mathrm{Rep}_k G$ denote the category of smooth representatons of G over k. When $G = \mathrm{GL}_n(\mathbb{Q}_p)$, locally profinite, we add to the definition being admissible, finite length, with action of the center $Z(G)$ by homotheties.

Acknowledgments

Contents

2   Recollection about Zábrádi's functor

Owing to the intended shortness of this note, we restrict our definitions and recollection to the minimum. The first ingredient of $\mathbf{V}_\Delta$ is the multivariable Fontaine equivalence proven in [Záb18b].

Let Δ be an (abstract for now) finite set.

Definition 2.1. When $\mathbb{F}$ is finite, we define the rings

$$E_\Delta^+ := \mathbb{F}[\![X_\alpha \,|\, \alpha \in \Delta]\!] \quad \text{and} \quad E_\Delta := E_\Delta^+[X_\Delta^{-1}],$$

where $X_\Delta = \prod X_\alpha$ and equips it with an $\mathbb{F}$-linear[1] action of

$$(\Phi_{\Delta, q} \times \Gamma_{\mathbb{Q}_p, \Delta}) := \prod_{\alpha \in \Delta} \varphi_\alpha^{\mathbb{N}} \times \prod_{\alpha \in \Delta} \mathbb{Z}_p^\times$$

that verifies

$$\forall \varphi = (\varphi_\alpha^{n_\alpha})_\alpha, \; \forall \beta \in \Delta, \; \varphi(X_\beta) = X_\alpha^{p^{n_\alpha}}$$

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[1] As $\mathbb{F}$ keeps track of the representations' coefficients, the action of $\Phi_{\Delta, p}$ must fix it.

$$\forall \gamma = (\gamma_\alpha)_\alpha \in \Gamma_{\mathbb{Q}_p, \Delta}, \ \forall \beta \in \Delta, \ \gamma(X_\beta) = (1 + X_\beta)^{\gamma_\beta} - 1.$$

In general, we define E_Δ^+ and E_Δ as the inductive limit of these rings over finite subfields of $\mathbb{F}$.

Let $E := \mathbb{F}_p((X))$. For each α, fix a copy $E_\alpha := \mathbb{F}_p((X_\alpha))$ of E, and an choose an extension of the identification to $E_\alpha^{\mathrm{sep}} \simeq E^{\mathrm{sep}}$, giving an identification $\mathcal{G}_{E_\alpha} \simeq \mathcal{G}_E$ then to $\mathcal{H}_{\mathbb{Q}_p}$. Each E_α has an embedding into E_Δ. We define

$$E_\Delta^{\mathrm{sep}} := E_\alpha^{\mathrm{sep}} \otimes_{E_\alpha} \left(\ldots \left(E_\beta^{\mathrm{sep}} \otimes_{E_\beta} E_\Delta \right) \right)$$

As in [Záb18b, §3.1], it comes with an action of $(\Phi_{\Delta,q} \times \mathcal{G}_{\mathbb{Q}_p, \Delta})$. The action of $\mathcal{H}_{\mathbb{Q}_p, \Delta}$ is given by the above isomorphisms $\mathcal{G}_{E_\alpha} \simeq \mathcal{H}_{\mathbb{Q}_p}$.

Thanks to [Záb18b, Lem. 3.6], we have the identity

$$(E_\Delta^{\mathrm{sep}})^{\Phi_{\Delta,p}} = \mathbb{F}.$$

We also define $\mathcal{H}_{\mathbb{Q}_p, \Delta} := \prod \mathcal{G}_{\mathbb{Q}_p(\mu_{p^\infty})} < \mathcal{G}_{\mathbb{Q}_p, \Delta}$, with quotient canoncially isomorphic to $\Gamma_{\mathbb{Q}_p, \Delta}$. Thanks to [Záb18b, Prop. 3.3], we have an $(\Phi_{\Delta,p} \times \Gamma_{\mathbb{Q}_p, \Delta})$-equivariant identification of rings:

$$(E_{\Delta^{\mathrm{sep}}})^{\mathcal{H}_{\mathbb{Q}_p, \Delta}} = E_\Delta.$$

Definition 2.2 (See [Mar24, §2.2]). For a monoid $\mathcal{S}$ acting on a ring R, we define $\mathrm{Mod}_{\mathrm{prj}}^{\mathrm{ét}}(\mathcal{S}, R)$ to be the category of finite projective R-module D of constant rank, equipped with a semilinear action of $\mathcal{S}$ such that

$$\forall s \in \mathcal{S}, \ R \otimes_{s,R} D \xrightarrow{\mathrm{Id} \otimes s} D$$

is an isomorphism.

After this setup, we can state the multivariable cyclotomic Fontaine equivalence.

Theorem 2.3. *The following pair of functors*

$$\mathbb{D}_\Delta : \qquad \mathrm{Rep}_{\mathbb{F}_p} \mathcal{G}_{\mathbb{Q}_p, \Delta} \rightleftarrows \mathrm{Mod}_{\mathrm{prj}}^{\mathrm{ét}}(\Phi_{\Delta,p} \times \Gamma_\Delta, E_\Delta) \qquad : \mathbb{V}_\Delta$$
$$V \mapsto (E_\Delta^{\mathrm{sep}} \otimes_{\mathbb{F}} V)^{\mathcal{H}_{\mathbb{Q}_p, \Delta}}$$
$$(E_\Delta^{\mathrm{sep}} \otimes_{E_\Delta} D)^{\Phi_{\Delta,p}} \leftarrowtail D$$

are well defined. They are quasi-inverse to one another, giving an equivalence of symmetric monoidal closed categories.

Remark 2.4. To pass from [Záb18b, Th. 3.15] to this theorem, one need three more ideas. The first one is that, for $\mathbb{F}$ finite, an object of $\mathrm{Rep}_{\mathbb{F}} \mathcal{G}_{\mathbb{Q}_p, \Delta}$ is an object of $\mathrm{Rep}_{\mathbb{F}_p} \mathcal{G}_{\mathbb{Q}_p, \Delta}$ with additional endomorphisms coding the external multiplication by $\mathbb{F}$. The same is true for $\mathrm{Mod}_{\mathrm{prj}}^{\mathrm{ét}}(\Phi_{\Delta,q} \times \mathcal{G}_{\mathbb{Q}_p, \Delta}, E_\Delta)$ compared to its version over $\mathbb{F}_p$. These structure are transfered on both side of Zábrádi's equivalence. This suffices when $\mathbb{F}$ is finite.

For general $\mathbb{F}$, one need to show that $\mathrm{Rep}_{\mathbb{F}} \mathcal{G}_{\mathbb{Q}_p, \Delta}$ and $\mathrm{Mod}_{\mathrm{prj}}^{\mathrm{ét}}(\Phi_{\Delta,q} \times \mathcal{G}_{\mathbb{Q}_p, \Delta}, E_\Delta)$ are colimits along base change of the analogous categories for $\mathbb{F}'$ varying among finite subfields of $\mathbb{F}$. For the Galois representations, its a general fact for a smooth representation of a compact group. For (φ, Γ)-modules, notice that $(\Phi_{\Delta,p} \times \Gamma_{\mathbb{Q}_p, \Delta})$ is topologically finitely generated and that its action on a multivariable (φ, Γ)-module is automatically continuous for some explicit topology (see [Mar25, Prop. 6.1.7]).

The last idea is that, according to [Záb18b, Prop. 2.2] [Mar25, Prop. 6.1.7], the projectivity condition is automatic.

We know prove a small useful fact that we did not see properly written anywhere.

Proposition 2.5. *Let $\Delta = \sqcup_{i \in I} \Delta_i$ be a partition of a finite set. Let $D_i \in \mathrm{Mod}_{\mathrm{prj}}^{\mathrm{ét}}(\Phi_{\Delta_i,p} \times \Gamma_{\mathbb{Q}_p, \Delta_i}, E_{\Delta_i})$. Then*

$$D := \otimes_{i \in I, E_\Delta}(E_\Delta \otimes_{E_{\Delta_i}} D_i)$$

belongs to $\mathrm{Mod}_{\mathrm{prj}}^{\mathrm{ét}}(\Phi_{\Delta,p} \times \Gamma_{\mathbb{Q}_p, \Delta}, E_\Delta)$ and naturally

$$\mathbb{V}_\Delta(D) \simeq \boxtimes_{i \in I} \mathbb{V}_{\Delta_i}(D_i).$$

Proof. View $(\Phi_{\Delta_i,p} \times \Gamma_{\mathbb{Q}_p,\Delta_i})$ as a quotient of $(\Phi_{\Delta,p} \times \Gamma_{\mathbb{Q}_p,\Delta})$. Then $E_{\Delta_i} \to E_\Delta$ is $(\Phi_{\Delta,p} \times \Gamma_{\mathbb{Q}_p,\Delta})$ for action by inflation on the first term. To prove that D is an étale projective module, combine [Mar24, Prop. 3.3 1), Prop 2.15 4)].

We first prove the identification for $\mathbb{F} = \mathbb{F}_p$ by expanding the strategy of [Mar25, Prop. 6.2.1]. By passing to the limit [Záb18b, Lem. 3.2] and using it for Δ, each Δ_i:

$$E_\Delta^{\mathrm{sep}} \simeq \bigotimes_{i \in I, \, E_\Delta} \left(E_{\Delta_i}^{\mathrm{sep}} \otimes_{E_{\Delta_i}} E_\Delta \right).$$

Hence

$$E_\Delta^{\mathrm{sep}} \otimes_{E_\Delta} D \simeq \bigotimes_{i \in I, \, E_\Delta^{\mathrm{sep}}} \left(E_\Delta^{\mathrm{sep}} \otimes_{E_{\Delta_i}^{\mathrm{sep}}} \left(E_{\Delta_i}^{\mathrm{sep}} \otimes_{E_{\Delta_i}} D_i \right) \right).$$

Combining [Záb18b, Prop. 3.7, Th. 3.15] we get comparison isomorphisms for each D_i, translating into

$$E_\Delta^{\mathrm{sep}} \otimes_{E_\Delta} D \simeq \bigotimes_{i \in I, \, E_\Delta^{\mathrm{sep}}} \left(E_\Delta^{\mathrm{sep}} \otimes_{E_{\Delta_i}^{\mathrm{sep}}} \left(E_{\Delta_i}^{\mathrm{sep}} \otimes_{\mathbb{F}_p} \mathbb{V}_{\Delta_i}(D_i) \right) \right) \simeq E_\Delta^{\mathrm{sep}} \otimes_{\mathbb{F}_p} \left(\underset{i \in I, \, \mathbb{F}_p}{\otimes} \mathbb{V}_{\Delta_i}(D_i) \right).$$

One can check that each of our isomorphisms is $(\Phi_{\Delta,p} \times \mathcal{G}_{E,\Delta})$-equivariant. Because $\boxtimes \mathbb{V}_{\Delta_i}(D_i)$ is free with trivial $\Phi_{\Delta,p}$-action (see [Záb18b, Lem. 3.6]), one obtains the result by taking $\Phi_{\Delta,p}$-invariants. $\qquad \square$

Corollary 2.6. *For $\Delta_1 \subset \Delta$ and $D_1 \in \mathrm{Mod}_{\mathrm{prj}}^{\mathrm{ét}} \left(\Phi_{\Delta_1,p} \times \Gamma_{\mathbb{Q}_p,\Delta_1}, E_{\Delta_1} \right)$,*

$$\mathbb{V}_\Delta \left(E_\Delta \otimes_{E_{\Delta_1}} D_1 \right) = \mathrm{Inf}_{\mathcal{G}_{\mathbb{Q}_p,\Delta_1}}^{\mathcal{G}_{\mathbb{Q}_p,\Delta}} \mathbb{V}_{\Delta_1}(D).$$

Proof. Apply for $I = \{1, 2\}$, $\Delta_2 = \Delta \backslash \Delta_1$ and $D_2 = E_{\Delta_2}$. $\qquad \square$

The other piece of Zábrádi's functor is an analogue of Colmez's $\mathbf{D}^\vee$. From now on, the set Δ is the canonical simple roots of the root system for the Lie group $\mathrm{GL}_n(\mathbb{Q}_p)$ obtained from the torus of diagonal matrices. It verifies $\#\Delta = n - 1$.

Definition 2.7. When $\mathbb{F}$ finite, the functor $\mathbf{D}_\Delta^\vee$ is the one introduced in [Záb18a]. It takes a smooth finite length[2] representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ over $\mathbb{F}$ with central character and produces an étale multivariable cyclotomic (φ, Γ)-module with Δ-variables.

In general, as any smooth representation Π of $\mathrm{GL}_n(\mathbb{Q}_p)$ over $\mathbb{F}$ is defined over a finite field[3] We can therefore apply the references over all its sufficiently large finite subfields., we define

$$\mathbf{D}_\Delta^\vee(\Pi) := \varprojlim_{\substack{\Pi'/\mathbb{F}_q \text{ s.t.} \\ \Pi = \mathbb{F} \otimes_{\mathbb{F}_q} \Pi'}} \mathbb{F} \otimes_{\mathbb{F}_q} \mathbf{D}_\Delta^\vee(\Pi')$$

with the above definition for $\mathbf{D}_\Delta^\vee(\Pi')$. One verifies that the category is filtered and all the transition maps in the limit are isomorphisms.

Let's combine it with the Fontaine equivalence.

Definition 2.8. Let $\delta_\Delta^\boxtimes$ be the character $\boxtimes_{i \in \Delta} \omega^{\sum_{j=n-i}^{n-1} j}$ of $\mathcal{G}_{\mathbb{Q}_p}^{n-1}$, whose restriction to the diagonal embedding is δ_{GL_n} from [Bre+21, p. 23]. We define $\mathbb{V}_\Delta^* := \underline{\mathrm{Hom}}(\mathbb{V}_\Delta, \delta_\Delta^\boxtimes)$. This could be considered as a good replacement for Tate duality.

Definition 2.9. We define $\mathbf{V}_\Delta := \mathbb{V}_\Delta^* \circ \mathbf{D}_\Delta^\vee$. It's a contravariant functor from the category $\mathrm{Rep}_\mathbb{F} \, \mathrm{GL}_n(\mathbb{Q}_p)$ to $\mathrm{Rep}_\mathbb{F} \, \mathcal{G}_{\mathbb{Q}_p,\Delta}$.

We call $\mathbf{V}_{\Delta,\xi}$ for the restriction of $\mathbf{V}_\Delta$ to the diagonal embedding.

Recall three important theorems about Zábrádi's functor.

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[2]As we don't know if all irreducible are admissible, note that it vanishes on non admissible representations.

[3]Use that $\mathrm{GL}_n(\mathbb{Q}_p)$ is generated by $\mathrm{GL}_n(\mathbb{Z}_p)$ and a finite number of elements.

Theorem 2.10. *[Th. C and G in [Záb18a]] The functor $\mathbf{D}_\Delta^\vee$ is right exact. It is exact on the subcategory* SP *of representations whose Jordan-Hölder factors are subquotients of principal series*[4]. *It is fully faithful on the subcategory* SP^0 *of representations whose Jordan-Hölder factors are irreducible principal series.*

Dual results holds for $\mathbf{V}_\Delta$ as both $\mathbb{V}_\Delta$ and $\underline{\mathrm{Hom}}$ are exact.

Theorem 2.11. *[Th. C in [JZ25]] If Π is irreducible smooth, then $\mathbf{V}_\Delta(\Pi)$ is either zero or irreducible.*

Theorem 2.12. *[Prop. 3.2 and Th. 3.5 in [Záb18a]] Let P be a standard parabolic of GL_n, whose Levi subgroup can be written as the diagonal $\prod_{1 \leq t \leq s} \mathrm{GL}_{m_s}$. Always chosing the root system associated with the standard torus, the intersection Δ_t of GL_{m_t}'s roots with Δ gives standard simple roots of GL_{m_t}.*

For all t, let Π_t be a smooth admissible representation of $\mathrm{GL}_{m_t}(\mathbb{Q}_p)$ over $\mathbb{F}$. We see $\boxtimes_t \Pi_t$ as a representation of $P^-(\mathbb{Q}_p)$ by inflation. Then,

$$\mathbf{D}_\Delta^\vee\Big(\mathrm{Ind}_{Q^-(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)}(\boxtimes_t \Pi_t)\Big) \simeq \bigotimes_{1 \leq t \leq s} \big(E_\Delta \otimes_{E_{\Delta_t}} \mathbf{D}_{\Delta_t}^\vee(\Pi_t)\big).$$

Definition 2.13. We define a canonical surjection $E_\Delta \twoheadrightarrow E$ as being the quotient by the ideal $(X_\alpha - X_\beta \,|\, \alpha, \beta \in \Delta)$, i.e. the identification of all variables to X. It is $(\varphi^\mathbb{N} \times \Gamma)$-equivariant for its action on E_Δ via the diagonal embedding into $(\Phi_{\Delta,p} \times \Gamma_{\mathbb{Q}_p,\Delta})$.

The tensor product of this canonical map with the identifications $E_\alpha^{\mathrm{sep}} \simeq E^{\mathrm{sep}}$ gives a surjection $E_\Delta^{\mathrm{sep}} \twoheadrightarrow E^{\mathrm{sep}}$. It is $(\varphi^\mathbb{N} \times \mathcal{G}_{\mathbb{Q}_p})$-equivariant for its action on E_Δ via the diagonal embedding into $(\Phi_{\Delta,p} \times \mathcal{G}_{\mathbb{Q}_p,\Delta})$.

Lemma 2.14. *There is a natural isomorphism of functors from $\mathrm{Mod}_{\mathrm{prj}}^{\text{ét}}\big(\Phi_{\Delta,p} \times \Gamma_{\mathbb{Q}_p,\Delta}, E_\Delta\big)$ to $\mathrm{Rep}_\mathbb{F}\,\mathcal{G}_{\mathbb{Q}_p}$*

$$\mathrm{Res}_{\mathcal{G}_{\mathbb{Q}_p}}^{\mathcal{G}_{\mathbb{Q}_p,\Delta}} \circ \mathbb{V}_\Delta \Rightarrow \mathbb{V}\left(E \otimes_{E_\Delta} -\right).$$

Here, the restriction is along the diagonal embedding and the tensor product is correctly defined using [Mar24, Prop. 3.3] for the $(\varphi^\mathbb{N} \times \Gamma)$-equivariant canonical map.

Proof. Factorising $E_\Delta \twoheadrightarrow E \hookrightarrow E^{\mathrm{sep}}$ through E_Δ^{sep}, we obtain a natural map

$$\big(E_\Delta^{\mathrm{sep}} \otimes_{E_\Delta} D\big)^{\Phi_{\Delta,p}} \to \Big(E^{\mathrm{sep}} \otimes_{E_\Delta^{\mathrm{sep}}} \big(E_\Delta^{\mathrm{sep}} \otimes_{E_\Delta} D\big)\Big)^{\varphi=\mathrm{Id}}.$$

By equivariance of this factorisation, it is a map of $\mathcal{G}_{\mathbb{Q}_p}$-representations, hence identifies with the predicted natural transformation.

To verify that it is a natural isomorphism, we can forget the action of $\mathcal{G}_{\mathbb{Q}_p}$. The isomorphism of comparison, proved together with multivariable cyclotomic Fontaine equivalence, implies that $E_\Delta^{\mathrm{sep}} \otimes_{E_\Delta} D \simeq E_\Delta^{\oplus \mathrm{rk} D}$ in $\mathrm{Mod}_{\mathrm{prj}}^{\text{ét}}(\Phi_{\Delta,p}, E_\Delta^{\mathrm{sep}})$. Hence, proving that the map constructed above is an isomorphism boils down to the case $D = E_\Delta$, i.e. to the equality $(E_\Delta^{\mathrm{sep}})^{\Phi_{\Delta,p}} = (E^{\mathrm{sep}})^{\varphi=\mathrm{Id}} = \mathbb{F}$ and the $\mathbb{F}$-linearity of all the maps in Définition 2.13. $\square$

Corollary 2.15. *For any smooth representation Π of $\mathrm{GL}_n(\mathbb{Q}_p)$, we have a natural injection $\mathbf{V}_{\Delta,\xi}(\Pi) \hookrightarrow \mathbf{V}_\xi(\Pi)$.*

Proof. Using[5] [Záb18a, Rem. 2 p. 13], there is a natural surjection

$$\mathbf{D}_\xi^\vee(\Pi) \twoheadrightarrow E \otimes_{E_\Delta} \mathbf{D}_\Delta^\vee(\Pi).$$

Let χ_{GL_n} be the 1-dimensional (φ, Γ)-module such that $\mathbb{V}(\chi_{\mathrm{GL}_n}) = \delta_{\mathrm{GL}_n}$. It is obtained by restriction to diagonal of the multivariable (φ, Γ)-module $\chi_\Delta^\boxtimes$ such that $\mathbb{V}_\Delta(\chi_\Delta^\boxtimes) = \delta_\Delta^\boxtimes$. We pass to duals and twist by χ_{GL_n}:

$$\underline{\mathrm{Hom}}(E \otimes_{E_\Delta} \mathbf{D}_\Delta^\vee(\Pi), \chi_{\mathrm{GL}_n}) \hookrightarrow \underline{\mathrm{Hom}}(\mathbf{D}_\xi^\vee, \chi_{\mathrm{GL}_n}).$$

By [Mar24, Prop. 3.3 3)], internal $\underline{\mathrm{Hom}}$ and base change commute, identifying the left part to $E \otimes_{E_\Delta} \underline{\mathrm{Hom}}(\mathbf{D}_\Delta^\vee(\Pi), \chi_\Delta^\boxtimes) \simeq E \otimes_{E_\Delta} \mathbf{V}_\Delta(\Pi)$. We apply Fontaine's functor $\mathbb{V}$ to the previous injection, still giving an injection thanks to its exactness. The left term is identified by Lemma 2.14 to $\mathbf{V}_{\Delta,\xi}(\Pi)$. The commutation of $\mathbb{V}$ to internal Hom identifies the right term with $\mathbf{V}_\xi(\Pi)$. $\square$

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[4]The principal series are $\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)}\chi$ for the standard Borel B and χ ranging over characters of the standard torus.

[5]And the finiteness proved in [JZ25].

3 Imprecise study of $\overline{L}^{\boxtimes}(\rho)$ and first failure

Our first result will rely on two ingredients: [JZ25, Th. C] and counting Jordan-Hölder components.

In this section, we fix an n-dimensional object ρ of $\mathrm{Rep}_{\mathbb{F}}\,\mathcal{G}_{\mathbb{Q}_p}$. We fix a composition serie for ρ and write the successive irreducible pieces $(\rho_l)_{1\le l \le k}$ of respective dimensions $\underline{n} = (n_l)$. In this situation, P_ρ is the parabolic whose Levi is the diagonal subgroup $M_{P_\rho} = \mathrm{GL}_{n_1} \times \ldots \times \mathrm{GL}_{n_k}$.

Definition 3.1. For such a tuple $\underline{n}$ and $i \ge 1$, call

$$\mathrm{Cut}_{i,\underline{n}} := \left\{ (j_l) \in \mathbb{N}_{\ge 1}^{[\![1,k]\!]} \,\Big|\, {\textstyle \sum j_l = i \atop \forall l,\, 1 \le i_l \le n_l} \right\}.$$

Lemma 3.2. *Let $1 \le i < n$. The representation $\Lambda^i \rho$ has more that $\#\mathrm{Cut}_{i,\underline{n}}$ Jordan-Hölder components.*

Proof. As our composition serie for ρ produces one for $\Lambda^i \rho$, the claim is equivalent to its analogue for $\rho^{\mathrm{ss}} = \oplus_l \rho_l$. For this, we have an isomorphism

$$\Lambda^i \rho^{\mathrm{ss}} \xleftarrow{\otimes_l \lambda_j v_{l,j} \mapsto \lambda_{l,j} v_{l,j}} \bigoplus_{(j_l)\in\mathrm{Cut}_{i,\underline{n}}} \bigotimes_{1\le l \le k} \left(\Lambda^{j_l} \rho_l \right)$$

with only non zero in the sum. It concludes. $\qquad\square$

Corollary 3.3. *The representation $\overline{L}^{\boxtimes}(\rho)$ has more that $\prod_{1\le i<n} \#\mathrm{Cut}_{i,\underline{n}}$ Jordan-Hölder components.*

Proof. Deduced from the fact that if V and W are representation of G and H with two (composition) series $(V_a)_{a\le A}$ and $(W_b)_{b\le B}$, then $(V_{a-1} \boxtimes W + V_a \boxtimes W_b)_{a,b\le A,B}$ with lexicographic order is a (composition) serie for $V \boxtimes W$. $\qquad\square$

Until the end of the section, suppose that ρ is generic and a good conjugate[6] in the sense of [Bre+21, §2.3.2].

Lemma 3.4. *The number of isotypic components of $\overline{L}^{\otimes}_{|Z_{M_{P_\rho}}}$ is smaller than $\prod_{1\le i<n} \#\mathrm{Cut}_{i,\underline{n}}$, strictly if ρ is not irreducible and $n > 2$.*

Proof. Call $(e_r)_{1\le r \le n}$ the canonical basis of the standard representation. The family $(\otimes_i (\wedge_{r\in I_i} e_r))_{(I_i)}$ for (I_i) ranging over in $\prod_{1\le i<n}\{I \subset [\![1,n]\!] \mid \#I = i\}$ forms a basis of $Z_{M_{P_\rho}}$-eigenvectors of $\overline{L}^{\otimes}$. Hence, the isotypic components are parametrised by

$$J_{\underline{n}} := \left\{ (\mu_l) \in \mathbb{N}^{[\![1,k]\!]} \,\Big|\, {\exists (I_i)_{1\le i<n} \text{ subsets of } [\![1,n]\!] \atop {\text{s.t. } \forall i,\, \#I_i=i \atop \text{and } \forall l,\, \sum_i \#(I_i\cap [\![\sum_{l'<l} n_{l'}+1, \sum_{l'\le l} n_{l'}]\!])=\mu_l}} \right\}.$$

There is a map

$$\prod_{1\le i<n} \mathrm{Cut}_{i,\underline{n}} \to J_{\underline{n}}, \quad (j_{i,l})_{i,l} \mapsto \left(\sum_i j_{i,l} \right)_l.$$

It is well defined because $(I_i = \cup_l [\![\sum_{l'<l} n_{l'}+1, \sum_{l'<l} n_{l'}+j_{i,l}]\!])$ witnesses that the image of $(j_{i,l})$ belongs to $J_{\underline{n}}$. It is also surjective, cause every (μ_l) with (I_i) witnessing its belonging to $J_{\underline{n}}$, is the image of $(\#(I_i \cap [\![\sum_{l'<l} n_{l'}+1, \sum_{l'\le l} n_{l'}]\!]))_{i,l}$.

This proves the first part, and the second amounts to finding a default of injectivity when ρ is reducible, i.e. $k \ge 2$ and $n > 2$.

If $k \ge 3$, we get inspiration from the example of $n = 3$ and $k = 3$, i.e. $P_\rho = B \subset \mathrm{GL}_3$. For this example, $e_1 \otimes (e_2 \wedge e_3)$ and $e_2 \otimes (e_1 \wedge e_3)$ are two eigenvectors for the same character $\mathrm{Diag}(x,y,z) \mapsto xyz$ of the torus, but they correspond to the two families of cuts $((1,0,0),(0,1,1))$ and $((0,1,0),(1,0,1))$. We mimicry this in the general situation. Take any family $(I_i)_{3\le i<n}$ and look at (μ_l) corresponding to

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[6]This is very mild asumption: genericity takes away multiplicities and the pathological extensions of characters already appearing for $\mathrm{GL}_2(\mathbb{Q}_p)$ (see [Col10, VII.4 Atomes de longueur 3]). Under this hypothesis, [Bre+21, §2.3.2] shows that good conjugates exist and all have similar notions of compatibility.

$(\{1\}, \{2,3\}, I_3, \dots, I_{n-1})$. Then $(\{2\}, \{1,3\}, I_3, \dots, I_{n-1})$ also witnesses the belonging of (μ_l) to $J_{\underline{n}}$. These two witnesses give two elements in the preimage of (μ_l).

If $k = 2$, there exists an l_0 such that $n_{l_0} \geq 2$. We get inspiration from the example of $n = 3$ and $(n_1, n_2) = (1, 2)$, i.e. P_ρ is the parabolic associated to $\mathbb{G}_m \times \mathrm{GL}_2$. In this situation, the same two vectors are eigenvectors for the same character $\mathrm{diag}(x, y, y) = xy^2$ of $Z_{M_{P_\rho}}$. We mimicry this in the general situation fixing l_0 as above and $l_1 \neq l_0$. Take any family $(I_i)_{3 \leq i < n}$ and look at (μ_l) corresponding to $(\{\sum_{l<l_0} n_l + 1\}, \{\sum_{l<l_1} n_l + 1, \sum_{l<l_0} n_l + 2\}, I_3, \dots, I_{n-1})$. Then, the family $(\{\sum_{l<l_1} n_l + 1\}, \{\sum_{l<l_0} n_l + 1, \sum_{l<l_0} n_l + 2\}, I_3, \dots, I_{n-1})$ also witnesses the belonging of (μ_l) to $J_{\underline{n}}$. Same conclusion. $\square$

Theorem 3.5. *Let $n \geq 3$ and ρ be a n-dimensional reducible object of $\mathrm{Rep}_{\mathbb{F}}\, \mathcal{G}_{\mathbb{Q}_p}$, which is generic and a good conjugate. Then any Π in $\mathrm{Rep}_{\mathbb{F}}\, \mathrm{GL}_n(\mathbb{Q}_p)$ that is compatible with $\widetilde{P}_\rho$ verifies*

$$\mathbf{V}_\Delta(\Pi) \not\simeq \overline{L}^{\boxtimes}(\rho).$$

Proof. Thanks to [Bre+21, Lem. 2.4.1.1 (i)], the number of Jordan-Hölder components of Π is precisely the number of isotypic components of $\overline{L}^{\boxtimes}_{|Z_{M_{P_\rho}}}$ i.e. $J_{\underline{n}}$. For each one of these, the conditions (i) and (iv) of [Bre+21, Def. 2.4.1.5] tells that there's a standard parabolic Q with Levi being the diagonal $M_Q = \mathrm{GL}_{m_1} \times \dots \times \mathrm{GL}_{m_s}$ and supersingular representations[7] Π_t of $\mathrm{GL}_{m_t}(\mathbb{Q}_p)$ such that the component is isomorphic to $\mathrm{Ind}_{Q^-(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)}(\boxtimes_t \Pi_t)$. As usual $\boxtimes_t \Pi_t$ is seen as a $Q^-(\mathbb{Q}_p)$-representation via inflation from M_Q. Thanks to Theorem 2.12, we have

$$\mathbf{D}_\Delta^\vee \left(\mathrm{Ind}_{Q^-(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)}(\boxtimes_t \Pi_t) \right) \simeq \bigotimes_{1 \leq t \leq s} \left(E_\Delta \otimes_{E_{\Delta_t}} \mathbf{D}_{\Delta_t}^\vee(\Pi_t) \right)$$

where Δ_t is set of simple positive roots of GL_{m_t} obtained by selection $\alpha \in \Delta$ wich factorise throught $T \twoheadrightarrow T_{\mathrm{GL}_{m_t}}$. We then use Proposition 2.5 to see that

$$(\mathbb{V}_\Delta \circ \mathbf{D}_\Delta^\vee) \left(\mathrm{Ind}_{Q^-(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)}(\boxtimes_t \Pi_t) \right) \simeq \underset{1 \leq t \leq s}{\boxtimes} (\mathbb{V}_{\Delta_t} \circ \mathbf{D}_{\Delta_t}^\vee)(\Pi_t).$$

In our situation, tensor product and dual commutes. Thus, $\mathbf{V}_\Delta(\mathrm{Ind}_{Q^-(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)}(\boxtimes \Pi_t))$ is a twist of the inflation $\mathrm{Inf}_{\prod_t \mathcal{G}_{\mathbb{Q}_p, \Delta_t}}^{\mathcal{G}_{\mathbb{Q}_p, \Delta}} \left(\boxtimes_t \mathbf{V}_{\Delta_t}(\Pi_t) \right)$ along the projection $\mathcal{G}_{\mathbb{Q}_p, \Delta} \to \prod \mathcal{G}_{\mathbb{Q}_p, \Delta_t}$. These individual $\mathbf{V}_{\Delta_t}(\Pi_t)$ are zero or irreducible thanks to [JZ25, Th. C]. Hence, so is the inflation of their boxproduct. We obtained that each Jordan-Hölder factor of Π is sent by $\mathbf{V}_\Delta$ on either zero or irreducible. Using right exactness of $\mathbf{V}_\Delta$, the number of Jordan-Hölder components of $\mathbf{V}_\Delta(\Pi)$ is less than $\#J_{\underline{n}}$.

Thanks to Corollay 3.3 and Lemma 3.4, this is less than the number of irreducible components of $\overline{L}^{\boxtimes}(\rho)$. $\square$

The same counting of Jordan-Hölder components can work for irreducible representations.

Theorem 3.6. *There exist infinitely many integers n such that there is an n-dimensional irreducible $\rho \in \mathrm{Rep}_{\mathbb{F}}\, \mathcal{G}_{\mathbb{Q}_p}$, with no supersingular $\Pi \in \mathrm{Rep}_{\mathbb{F}}\, \mathrm{GL}_n(\mathbb{Q}_p)$ verifying $\mathbf{V}_\Delta(\Pi) \simeq \overline{L}^{\boxtimes}(\rho)$.*

Proof. By Theorem 2.11, any $\mathbf{V}_\Delta(\Pi)$ for Π supersingular is either zero or irreducible. As soon as $\Lambda^i \rho$ for some $i \geq 2$, then $\overline{L}^{\boxtimes}(\rho)$ is reducible. Hence, we only need to find a family of representations ρ with unbounded dimensions for which $\Lambda^2 \rho$ is reducible.

Recall from local class field theory that

$$\mathcal{G}_{\mathbb{Q}_p} / \mathcal{I}_{\mathbb{Q}_p}^{\mathrm{wild}} \simeq \left(\varprojlim_{r \geq 1,\ \mathrm{norm\ map}} \mathbb{F}_{p^r}^\times \right) \rtimes \varphi^{\widehat{\mathbb{Z}}}.$$

Take any prime ℓ with $\ell \nmid p(p^2 - 1)(p^3 - 1)$ and $p \nmid (\ell - 1)$. They are infinitely many such thanks to Dirichlet prime number theorem. As $\ell | (p^r - 1)$ for some r, we can identify $\mathbb{Z}/\ell\mathbb{Z}$ as a quotient of

---

[7]This is equivalent to $\boxtimes_t \Pi_t$ being supersingular.

$\lim_{\leftarrow} \mathbb{F}_r^\times$ by a characteristic subgroup. Consider the action of $\varphi^{\widehat{\mathbb{Z}}}$ by conjugation on this quotient. It acts via multiplication by some element in $(\mathbb{Z}/\ell\mathbb{Z})^\times$. Quotienting again by the kernel of this action constructs a quotient Q_ℓ of $\mathcal{G}_{\mathbb{Q}_p}/\mathcal{I}_{\mathbb{Q}_p}^{\mathrm{wild}}$ isomorphic to some $G_\ell := (\mathbb{Z}/\ell\mathbb{Z}) \rtimes K_\ell$ where $K_\ell \lhd (\mathbb{Z}/\ell\mathbb{Z})^\times$. Remark that $\#K_\ell$ is the order of p in $(\mathbb{Z}/\ell\mathbb{Z})^\times$. For any d, the primes ℓ stop dividing $p^d - 1$ at some point, hence $\#K_\ell \xrightarrow{\ell \to +\infty} +\infty$.

We begin by looking more precisely into irreducible representations of Q_ℓ over $\overline{\mathbb{F}_p}$. Its conjugacy classes are the sets $\{\mathrm{Id}\}$, $aK_\ell \times \{1\}$ for $a \in (\mathbb{Z}/\ell\mathbb{Z})\backslash\{0\}$ and $(\mathbb{Z}/\ell\mathbb{Z}) \times \{x\}$ for $x \in K_\ell\backslash\{1\}$: there are $\#K_\ell + (l-1)/\#K_\ell$ of them. We also have $\#K_\ell$ irreducible non isomorphic 1-dimensional representations, obtained by inflation from K_ℓ. Thanks to our hypothesis on ℓ, we have $p \nmid \#G_\ell$, hence we can apply Frobenius theorem; writing m for the maximum dimension of an irreducible representation of Q_ℓ over $\overline{\mathbb{F}_p}$, we get

$$\#K_\ell + (l-1)m^2/(\#K_\ell) \geq \#Q_\ell = \ell(\#K_\ell)$$

then $m \geq \#K_\ell$. Pick one irreducible of dimension m. The (finite) sum of its distinct conjugates by $\mathcal{G}_{\mathbb{F}}$ can be descended to an irreducible representation over $\mathbb{F}$, of maximal dimension among such irreducibles. Its dimension is greater than $\#K_\ell$. We call such representation σ_ℓ. Because $\ell \nmid (p^2 - 1)(p^3 - 1)$, we have $\#K_\ell \geq 4$. As $\dim \Lambda^2 \sigma_\ell = \dim \sigma_\ell(\dim \sigma_\ell - 1)/2 > \dim \sigma_l$, the representation $\bigwedge^2 \sigma_\ell$ can't be irreducible. The family of $\mathrm{Inf}_{Q_\ell}^{\mathcal{G}_{\mathbb{Q}_p}} \sigma_\ell$ testifies that our infinite number of prime ℓ proves the theorem. $\qquad\square$

4 Setup for the GL_3-toy case

Our toy example for studying a "weak compatibility" hypothesis is a 3-dimensional representation ρ of $\mathcal{G}_{\mathbb{Q}_p}$ of length 3. Suppose it is generic and good conjugate, which means here that ρ can be written

$$\begin{pmatrix} \chi_1 & \delta_a & \epsilon \\ & \chi_2 & \delta_b \\ & & \chi_3 \end{pmatrix}$$

with all $\chi_i \chi_j^{-1} \notin \{1, \omega^{\pm 1}\}$.

The first thing we do is analyse the shape of $\overline{L}^{\boxtimes}(\rho)$ and $\overline{L}^{\otimes}(\rho)$.

Lemma 4.1. *The representation $\Lambda^2 \rho$ also has a length 3 and a composition serie given by*

$$\chi_1\chi_2 - \chi_1\chi_3 - \chi_2\chi_3.$$

Its socle (resp. its cosocle) is of same dimension as the cosocle (resp. the socle) of ρ.

Proof. In the basis $\{e_1 \wedge e_2,\ e_1 \wedge e_3,\ e_2 \wedge e_3\}$ the matrix representation is

$$\begin{pmatrix} \chi_1\chi_2 & \chi_1\delta_b & \delta_a\delta_b - \alpha\chi_2 \\ & \chi_1\chi_3 & \delta_a\chi_3 \\ & & \chi_2\chi_3 \end{pmatrix}.$$

For the statement about socle, consider that $\mathrm{Vect}(e_1 \wedge e_2, e_1 \wedge e_3)$ is semisimple iff δ_b is trivial, hence iff the cosocle of ρ is of dimension at least 2. Similar arguments for the other subquotients finish the proof. $\qquad\square$

Proposition 4.2. *The representation $\rho \boxtimes (\Lambda^2 \rho)$ of $(\mathcal{G}_{\mathbb{Q}_p} \times \mathcal{G}_{\mathbb{Q}_p})$ has a composition serie given by*

$$\chi_1 \boxtimes \chi_2\chi_3$$

The diagram (Loewy/Hasse structure):

$$
\begin{array}{ccccccc}
& & & \chi_1 \boxtimes \chi_2\chi_3 & & & \\
& & \chi_1 \boxtimes \chi_1\chi_3 & & \chi_2 \boxtimes \chi_2\chi_3 & & \\
\chi_1 \boxtimes (\chi_1\chi_2) & & \chi_2 \boxtimes \chi_1\chi_3 & & & \chi_3 \boxtimes \chi_2\chi_3 \\
& \chi_2 \boxtimes (\chi_1\chi_2) & & \chi_3 \boxtimes \chi_1\chi_3 & & \\
& & \chi_3 \boxtimes \chi_1\chi_2 & & & \\
\end{array}
$$
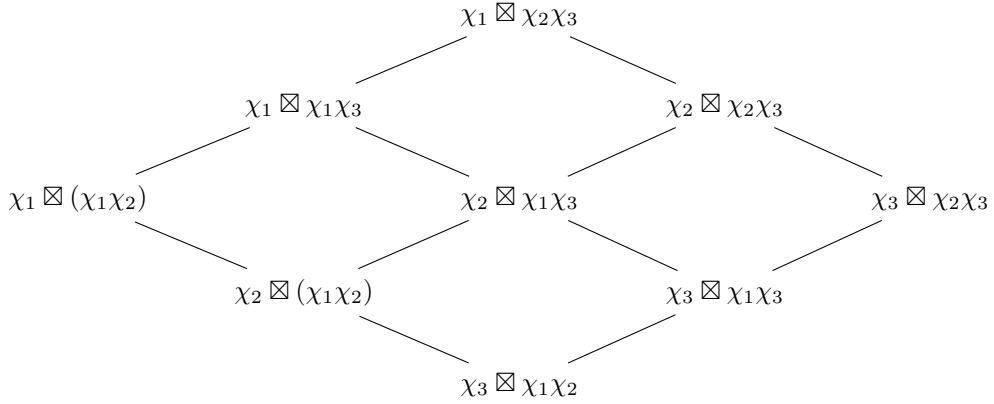
When ρ has Loewy length 3 (resp. 2, resp. is semisimple), it has Loewy length 5 (resp. 3, resp. 1) with dimensions of the graded pieces $(1,2,3,2,1)$ (resp. $(2,5,2)$, resp. 9).

Proof. Consider the convolution formula for boxproducts of smooth representations of finite dimension. One could also write down explictely the exterior tensor product of matrices[8]. □

We write $\overline{L}^{\boxtimes}(\rho)^{\mathrm{ord}}$ for the subrepresentation of $\overline{L}^{\boxtimes}(\rho)$ having $(\chi_1 \boxtimes \chi_1\chi_2)$, $(\chi_1 \boxtimes \chi_1\chi_3)$ and $(\chi_2 \boxtimes \chi_1\chi_2)$ as Jordan-Hölder factors. In case ρ is of Loewy length 3 (maximally non split), it's $\mathrm{soc}^2_{\mathcal{G}_{\mathbb{Q}_p} \times \mathcal{G}_{\mathbb{Q}_p} \times \mathbb{Q}_p^\times} \overline{L}^{\boxtimes}(\rho)$. We will switch the symbol $\boxtimes$ to an $\otimes$ for their restriction to the diagonal embedding of $\mathcal{G}_{\mathbb{Q}_p}$.

Remark 4.3. The representation $\overline{L}^{\otimes}(\rho)^{\mathrm{ord}}$ coincide with $(\overline{L}^{\otimes}_{|B})^{\mathrm{ord}} \circ \rho$ where the first is defined in [BH15, §2.5]. See also [Bre15] where this composition appears.

Corollary 4.4. *If $p \neq 2$, the $\mathcal{G}_{\mathbb{Q}_p}$-representation*

$$\mathrm{Res}^{(\mathcal{G}_{\mathbb{Q}_p} \times \mathcal{G}_{\mathbb{Q}_p})}_{\mathcal{G}_{\mathbb{Q}_p}}(\overline{L}^{\boxtimes}(\rho)/\overline{L}^{\boxtimes}(\rho)^{\mathrm{ord}})$$

has a composition serie like

$$
\begin{array}{ccc}
\det\rho \;\text{——}\; \chi_2^2\chi_3 & & \\
& \searrow & \\
\det\rho & & \chi_2\chi_3^2 \\
& \nearrow & \\
\det\rho \;\text{——}\; \chi_1\chi_3^2 & & \\
\end{array}
$$
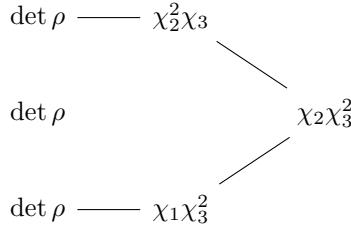
The columns give the socle filtration when ρ is of Loewy length 3.

Proof. In the basis $\{e_1\otimes(e_2\wedge e_3), e_2\otimes(e_1\wedge e_3), e_3\otimes(e_1\wedge e_2), e_2\otimes(e_2\wedge e_3), e_3\otimes(e_1\wedge e_3), e_3\otimes(e_2\wedge e_3)\}$, the matrix looks like

$$
\begin{pmatrix}
\det\rho & 0 & 0 & \chi_2\chi_3\delta_a & 0 & * \\
& \det\rho & 0 & \chi_2\chi_3\delta_a & \chi_1\chi_3\delta_b & * \\
& & \det\rho & 0 & \chi_1\chi_3\delta_b & * \\
& & & \chi_2^2\chi_3 & 0 & \chi_2\chi_3\delta_b \\
& & & & \chi_1\chi_3^2 & \chi_3^2\delta_a \\
& & & & & \chi_2\chi_3^2
\end{pmatrix}
$$

---

[8]Evaluated at a couple !

We base change the first three vectors via a matrix $\left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{smallmatrix}\right)$, invertible as soon as $p \neq 2$. We get the matrix

$$\begin{pmatrix} \det\rho & 0 & 0 & \chi_2\chi_3\delta_a & 0 & * \\ & \det\rho & 0 & 0 & 0 & * \\ & & \det\rho & 0 & \chi_1\chi_3\delta_b & * \\ & & & \chi_2^2\chi_3 & 0 & \chi_2\chi_3\delta_b \\ & & & & \chi_1\chi_3^2 & \chi_3^2\delta_a \\ & & & & & \chi_2\chi_3^2 \end{pmatrix}$$

which gives the announced results.

$\square$

Remark 4.5. Be careful that socle filtration and restriction do not commute: for instance, there's always a copy of $\det\rho$ in $\mathrm{soc}^2_{\mathcal{G}_{\mathbb{Q}_p}}(\overline{L}^{\otimes}(\rho))$.

From now on, assume that ρ has Loewy length 3. In this situation, the representation predicted by [Bre+21, §2.4.3 Ex. 2] is

$$
\begin{array}{ccccccc}
 & & \mathrm{PS}_{\chi_2,\chi_1,\chi_3} & & \mathrm{PS}_{\chi_2,\chi_3,\chi_1} & & \\
 & \diagup & & \diagdown \quad \diagup & & \diagdown & \\
\mathrm{PS}_{\chi_1,\chi_2,\chi_3} & & & \mathrm{SS} & & & \mathrm{PS}_{\chi_3,\chi_2,\chi_1} \\
 & \diagdown & & \diagup \quad \diagdown & & \diagup & \\
 & & \mathrm{PS}_{\chi_1,\chi_3,\chi_2} & & \mathrm{PS}_{\chi_3,\chi_1,\chi_2} & &
\end{array}
\qquad (*1)
$$

with θ being the product of simple roots, the representations $\mathrm{PS}_{\chi'}$ defined as $\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)} \chi'(\varepsilon^{-1} \circ \theta)$ and all extensions being non split.

Definition 4.6. For such a representation Π compatible with $\widetilde{P}_\rho$, we note Π^{ord} (resp. Π_{SS}, resp. Π_{ord}) the subrepresentation $\mathrm{soc}^2_{\mathrm{GL}_3(\mathbb{Q}_p)}\Pi$ (resp. $\mathrm{soc}^3_{\mathrm{GL}_3(\mathbb{Q}_p)}\Pi$, resp. $\Pi/\mathrm{soc}^3_{\mathrm{GL}_3(\mathbb{Q}_p)}\Pi$).

Remark 4.7. Breuil and Herzig constructed in [BH15] a representation $\Pi(\rho)^{\mathrm{ord}}$ that is conjectured to be the maximal subrepresentation of some $\Pi(\rho)$ constructed by global methos, with only principal serie Jordan-Hölder component. It coincides with Π^{ord} in our setting for any Π compatible with P_ρ.

We first explain why Zábrádi's functor applied to Π^{ord} recovers $\overline{L}^{\boxtimes}(\rho)^{\mathrm{ord}}$.

Definition 4.8. For a smooth character $\chi : T(\mathbb{Q}_p) \to \overline{\mathbb{F}_p}^\times$, we view it as a character of the product $(\mathrm{W}_{\mathbb{Q}_p}^{\mathrm{ab}} \times \mathrm{W}_{\mathbb{Q}_p}^{\mathrm{ab}} \times \mathbb{Q}_p^\times)$ via

$$(\mathrm{W}_{\mathbb{Q}_p}^{\mathrm{ab}} \times \mathrm{W}_{\mathbb{Q}_p}^{\mathrm{ab}} \times \mathbb{Q}_p^\times) \xrightarrow{\sim} T(\mathbb{Q}_p), \quad (g, h, x) \mapsto \mathrm{diag}(\mathrm{Art}^{-1}(gh)x, \mathrm{Art}^{-1}(h)x, x).$$

We restrict to $(\mathrm{W}_{\mathbb{Q}_p}^{\mathrm{ab}} \times \mathrm{W}_{\mathbb{Q}_p}^{\mathrm{ab}})$, extend by smoothness and inflate, obtaining a character χ_Δ of $(\mathcal{G}_{\mathbb{Q}_p} \times \mathcal{G}_{\mathbb{Q}_p})$.

Lemma 4.9. *Let $\chi : T(\mathbb{Q}_p) \to \overline{\mathbb{F}_p}^\times$ be a smooth character, seen by inflation as a character of $B^-(\mathbb{Q}_p)$. Then,*

$$\mathbf{V}_\Delta\left(\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)} \chi\right) = \chi_\Delta \otimes \delta_\Delta^{\boxtimes}.$$

Proof. For $G = T$, the monoid T_+ of Zábrádi is $T(\mathbb{Q}_p)$ and the functor $\mathbf{D}_\emptyset^\vee$ is the dual on characters of $T(\mathbb{Q}_p)$. The result follows from [Záb18a, Th. 3.5] and our expression of $\mathbb{V}_\Delta^*$. $\square$

Proposition 4.10. *The functor $\mathbf{V}_\Delta$ establishes an isomorphism between*

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\left(\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)} s_\alpha(\chi)(\varepsilon^{-1} \circ \theta) \oplus \mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)} s_\beta(\chi)(\varepsilon^{-1} \circ \theta), \mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)} \chi(\varepsilon^{-1} \circ \theta)\right)$$

where the extension group is taken in $\mathrm{Rep}_{\overline{\mathbb{F}_p}} \mathrm{GL}_3(\mathbb{Q}_p)$, and

$$\mathrm{Ext}^1_{(\mathcal{G}_{\mathbb{Q}_p} \times \mathcal{G}_{\mathbb{Q}_p})}\big((\chi_2 \boxtimes (\chi_1\chi_2)) \oplus (\chi_1 \boxtimes (\chi_1\chi_3)), (\chi_1 \boxtimes (\chi_1\chi_2))\big)$$

where the extension group is taken in $\mathrm{Rep}_{\overline{\mathbb{F}_p}} \mathcal{G}_{\mathbb{Q}_p}$.

The functor $\mathbf{V}_{\Delta,\xi}$ *establishes a bijection between the same* $\mathrm{GL}_3(\mathbb{Q}_p)$-*Ext-group and*

$$\mathrm{Ext}^1_{\mathcal{G}_{\mathbb{Q}_p}}(\chi_1\chi_2^2 \oplus \chi_1^2\chi_3, \chi_1^2\chi_2).$$

The same kind of result holds for

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p),Z}\left(\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)}\chi^s(\varepsilon^{-1}\circ\theta), \mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)}s_\beta(\chi^s)(\varepsilon^{-1}\circ\theta) \oplus \mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)}s_\alpha(\chi^s)(\varepsilon^{-1}\circ\theta)\right)$$

with χ^s *being the conjugate by the element of maximal length in the Weyl group of* GL_3.

Proof. The previous Lemma implies that

$$\mathbf{V}_\Delta(\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)}\chi(\varepsilon^{-1}\circ\theta)) = (\chi(\varepsilon^{-1}\circ\theta))_\Delta \otimes \delta_\Delta^{\boxtimes} = \chi_1 \boxtimes (\chi_1\chi_2)$$

and similarly for the others. Moreover, the genericity hypothesis and [Oll06, Th. 4] prove that all these inductions lie in SP^0. With this remark, it is a reformulation of [JZ25, Cor. 3.3.4].

Let's give the main ideas. First, the map is injective thanks to the faithfulness in Theorem 2.10. For essential surjectivity, we can do it independantly for the two extensions. Pick the first one. Thanks to [JZ25, Prop. 3.3.2], this extension must be trivial restricted to $\{\mathrm{Id}\} \times \mathcal{G}_{\mathbb{Q}_p}$. Genericity tell that the extension group is of dimension 1. Therefore [Hau16, Th. 1.1 ii)] produces a representation induced from the $\mathbb{Q}_p$-points of the parabolic P_α^- with Levi $\mathrm{GL}_2 \times \mathbb{G}_m$ which has the correct image. $\square$

Remark 4.11. It is an improvement for Zábrádi's functor of [Bre15, Th. 1.1] in the situation $G = \mathrm{GL}_3$.

5 Representations results and finer failure on our toy example

We keep our representations ρ of dimension 3 and Loewy length 3.

Definition 5.1. We say that Π is *weakly compatible with* ρ if it is compatible with $\widetilde{P}_\rho$ and[9] if $\mathbf{V}_\xi(\Pi) \simeq \overline{L}^\otimes(\rho)$.

Lemma 5.2. *Let* G *be a profinite group and* k *an algebraically closed field. Let* V *and* W *be irreducible objects in* $\mathrm{Rep}_k G$ *and* χ *a* 1-*dimensional character. Then* $\chi^{\oplus 2}$ *can't be a subrepresentation of* $V \otimes W$

Proof. As everything is finite dimensional, we have

$$\begin{aligned}
\mathrm{Hom}_{k[G]}(\chi, V \otimes W) &\simeq \mathrm{Hom}_{k[G]}\left((V \otimes W)^\vee, \chi^{-1}\right) \\
&\simeq \mathrm{Hom}_{k[G]}\left(V^\vee \otimes W^\vee, \chi^{-1}\right) \\
&\simeq \mathrm{Hom}_{k[G]}\left(V^\vee, W \otimes \chi^{-1}\right)
\end{aligned}$$

As V is irreducible, so is $V^\vee$. As W is irreducible, so is $W \otimes \chi^{-1}$. Hence, Schur's lemma says that this $\mathrm{Hom}_{k[G]}$ is at most of dimension 1. It concludes. $\square$

Lemma 5.3. *Let* V *be a representation of* $\mathcal{G}_{\mathbb{Q}_p}$ *over* $\mathbb{F}$ *such that* $\mathcal{I}_{\mathbb{Q}_p}^{\mathrm{wild}}$ *acts trivially. Any subrepresentation of* V *that is an extension of two distinct characters is semisimple.*

Proof. Fix such subrepresentation W. As any character is trivial restricted to the pro-p-group $\mathcal{I}_{\mathbb{Q}_p}^{\mathrm{wild}}$, one can twist and suppose that W is an extension of 1 by χ, hence lives in $H^1(\mathcal{G}_{\mathbb{Q}_p}/\mathcal{I}_{\mathbb{Q}_p}^{\mathrm{wild}}, \chi)$. We only need to prove that this group is zero as soon as $\chi \not\equiv 1$. We once again use that

$$\mathcal{G}_{\mathbb{Q}_p}/\mathcal{I}_{\mathbb{Q}_p}^{\mathrm{wild}} \simeq \left(\varprojlim_{r \geq 1,\,\mathrm{norm\ map}} \mathbb{F}_{p^r}^\times\right) \rtimes \varphi^{\widehat{\mathbb{Z}}}$$

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[9]This is condition (i) in [Bre+21, Def. 2.4.2.7] but only for $\Pi' = \Pi$.

with the big limite identified to $\mathcal{I}_{\mathbb{Q}_p}/\mathcal{I}_{\mathbb{Q}_p}^{\mathrm{wild}}$.

The inflation-restriction sequence for $\mathcal{K} := (\mathcal{I}_{\mathbb{Q}_p}/\mathcal{I}_{\mathbb{Q}_p}^{\mathrm{wild}}) \rtimes \varphi^{\prod_{l \neq p} \mathbb{Z}_l}$ gives

$$0 \to H^1(\varphi^{\mathbb{Z}_p}, \chi^{\mathcal{K}}) \to H^1(\mathcal{G}_{\mathbb{Q}_p}/\mathcal{I}_{\mathbb{Q}_p}^{\mathrm{wild}}, \chi) \to H^1(\mathcal{K}, \chi_{|\mathcal{K}})$$

As $\mathcal{K}$ is pro-(prime to p), the last bit vanishes. By examining the orders $\chi_{|\varphi^{\mathbb{Z}_p}} \equiv 1$, hence $\chi \not\equiv 1$ implies that $\chi_{|\mathcal{K}} \not\equiv 1$. Now, $\chi_{|\mathcal{K}} \not\equiv 1$, cause and we now that $\chi \not\equiv 1$. We get $\chi^{\mathcal{K}} = \{0\}$ and the first term also vanishes. $\qquad \square$

Corollary 5.4. *Let V be an irreducible object in $\mathrm{Rep}_{\overline{\mathbb{F}_p}}(\mathcal{G}_{\mathbb{Q}_p} \times \mathcal{G}_{\mathbb{Q}_p})$. Any $W \subset \mathrm{Res}_{\mathcal{G}_{\mathbb{Q}_p}}^{(\mathcal{G}_{\mathbb{Q}_p} \times \mathcal{G}_{\mathbb{Q}_p})} V$ which is an extension of two distincts characters is semisimple.*

Proof. The representation V must be written $V_1 \boxtimes V_2$ where V_i are an irreducible objects in $\mathrm{Rep}_{\overline{\mathbb{F}_p}} \mathcal{G}_{\mathbb{Q}_p}$. It is a general fact that any normal pro-p-subgroup of a profinite group acts trivially on any irreducible representation over a (inductive limite) of characteristic p fields. Both V_1 and V_2 factorise by $\mathcal{I}_{\mathbb{Q}_p}^{\mathrm{wild}}$. Thus, Lemma 5.3 applied to $\mathrm{Res}_{\mathcal{G}_{\mathbb{Q}_p}}^{(\mathcal{G}_{\mathbb{Q}_p} \times \mathcal{G}_{\mathbb{Q}_p})} V$ concludes. $\qquad \square$

We're ready for our second result.

Proposition 5.5. *Let ρ belong to $\mathrm{Rep}_{\overline{\mathbb{F}_p}} \mathcal{G}_{\mathbb{Q}_p}$, of length 3 and Loewy length 3, generic and good conjugate. Let Π be weakly compatible with ρ.*

1. *We have $\dim \mathbf{V}_\Delta(\Pi) \leq 5$.*

2. *We have $\overline{L}^{\boxtimes}(\rho)^{\mathrm{ord}} = \mathbf{V}_\Delta(\Pi^{\mathrm{ord}}) \hookrightarrow \mathbf{V}_\Delta(\Pi)$.*

3. *If $p \neq 2$, we have $\dim \mathbf{V}_\Delta(\Pi) \leq 4$, if and only if $\mathbf{V}_\Delta(\Pi) \to \mathbf{V}_\Delta(\Pi^{\mathrm{ord}})$ is non zero.*

The first point in particular implies that $\mathbf{V}_{\Delta,\xi}(\Pi) \not\simeq \overline{L}^{\boxtimes}(\rho)$.

Proof. We have an exact sequence

$$0 \to \mathbf{V}_\Delta(\Pi^{\mathrm{ord}}) \to \mathbf{V}_\Delta(\Pi) \to \mathbf{V}_\Delta(\Pi/\Pi^{\mathrm{ord}}).$$

In particular, thanks to weak compatibility and Corollary 2.15, we have $\mathbf{V}_{\Delta,\xi}(\Pi^{\mathrm{ord}}) \hookrightarrow \overline{L}^{\boxtimes}(\rho)$. Using the second bijection of Proposition 4.10, $\mathbf{V}_{\Delta,\xi}(\Pi^{\mathrm{ord}})$ is an extension of two distinct characters by a third. Each character appearing only once (thanks to genericity) in $\mathrm{soc}_{\mathcal{G}_{\mathbb{Q}_p}}^2 \overline{L}^{\boxtimes}(\rho)$, and more precisely they appear in its subrepresentation $\overline{L}^{\boxtimes}(\rho)^{\mathrm{ord}}$. It yields $\mathbf{V}_{\Delta,\xi}(\Pi^{\mathrm{ord}}) \simeq \overline{L}^{\boxtimes}(\rho)^{\mathrm{ord}}$.

Consider $V := \mathbf{V}_{\Delta,\xi}(\Pi)/\mathbf{V}_{\Delta,\xi}(\Pi^{\mathrm{ord}})$. Weak compatibility and the above isomorphism show that it is a subrepresentation of $\overline{L}^{\boxtimes}(\rho)/\overline{L}^{\boxtimes}(\rho)^{\mathrm{ord}}$. Thus, $\mathrm{soc}_{\mathcal{G}_{\mathbb{Q}_p}}(V) \hookrightarrow \mathrm{soc}_{\mathcal{G}_{\mathbb{Q}_p}}(\overline{L}^{\boxtimes}(\rho)/\overline{L}^{\boxtimes}(\rho)^{\mathrm{ord}})$ which is a direct sum of $(\det \rho)$. Left exactness of $\mathbf{V}_{\Delta,\xi}$ that it is a subrepresentation of $\mathbf{V}_{\Delta,\xi}(\Pi/\Pi^{\mathrm{ord}})$. This last one lives in an exact sequence

$$0 \to \mathbf{V}_{\Delta,\xi}(\Pi_{\mathrm{SS}}) \to \mathbf{V}_{\Delta,\xi}(\Pi/\Pi^{\mathrm{ord}}) \to \mathbf{V}_{\Delta,\xi}(\Pi_{\mathrm{ord}}).$$

Using the last bit of Proposition 4.10 to compute $\mathbf{V}_{\Delta,\xi}(\Pi_{\mathrm{ord}})$, $\det \rho$ is not one of its Jordan-Hölder factors, yielding that $\mathrm{soc}_{\mathcal{G}_{\mathbb{Q}_p}} V$ is a subrepresentation of $\mathbf{V}_{\Delta,\xi}(\Pi_{\mathrm{SS}})$. Theorem 2.11 says that $\mathbf{V}_{\Delta,\xi}(\Pi_{\mathrm{SS}})$ is the restriction of an irreducible representation of $(\mathcal{G}_{\mathbb{Q}_p} \times \mathcal{G}_{\mathbb{Q}_p})$; Lemma 5.2 proves that $\mathrm{soc}_{\mathcal{G}_{\mathbb{Q}_p}} V$ is one dimensional. As V is a subrepresentation of $\overline{L}^{\boxtimes}(\rho)/\overline{L}^{\boxtimes}(\rho)^{\mathrm{ord}}$, the description in Corollary 4.4 proves that it is at most 2-dimensional which finishes to prove the first claim.

It also proves that if $\dim V = 2$, V is an extension of two distincts (thanks to genericity) characters. Using Corollary 5.4, it can't be a subrepresentation of $\mathbf{V}_{\Delta,\xi}(\Pi_{\mathrm{SS}})$, which proves the third claim.

Finally, use simultaneously the two first isomorphisms of Proposition 4.10 to show that $\mathbf{V}_{\Delta,\xi}(\Pi^{\mathrm{ord}}) = \overline{L}^{\boxtimes}(\rho)^{\mathrm{ord}}$ implies $\mathbf{V}_\Delta(\Pi^{\mathrm{ord}}) = \overline{L}^{\boxtimes}(\rho)^{\mathrm{ord}}$. $\qquad \square$

Remark 5.6. We used nothing more that weak compatibility. With (stronger) compatibility in the sense of [Bre+21], we would obtain directly that $\mathbf{V}_{\Delta,\xi}(\Pi_{\mathrm{SS}})$ is 1-dimensional. I don't see how it may improves the results about dimensions.

Remark 5.7. Be careful that Lemma 5.2 breaks for $\#\Delta \geq 3$.

We could improve the weak compatibility by replacing $\mathbf{V}_\xi(\Pi) \simeq \overline{L}^{\otimes}(\rho)$ by $\mathbf{V}_\Delta(\Pi) \subset \overline{L}^{\boxtimes}(\rho)$. This also give precise bounds of the dimension. For ρ of dimension, length and Loewy length 4, it gives $\dim \mathbf{V}_\Delta(\Pi) \leq 9$.

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