Implications of Breuil-Herzig-Hu-Morra-Schraen's conjectures on Zábrádi's functor

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Abstract. — Let ρ be an n-dimensional representation of $\mathcal{G}_{\mathbb{Q}_p}$ over $\overline{\mathbb{F}_p}$. When ρ is generic and a good conjugate, the article [Bre+21] introduces the notion of an admissible representation Π of $\mathrm{GL}_n(\mathbb{Q}_p)$ compatibility with ρ. In *loc. cit.*, the five authors also question whether there exists some Π compatible with ρ from which Zábrádi's functor $\mathbf{V}_\Delta$(see [JZ25]) recovers an explicit representation $\overline{L}^{\boxtimes}(\rho)$ of $\mathcal{G}_{\mathbb{Q}_p}^{n-1}$, constructed from ρ. We give a range of results, for an arbitrary Π verifying some "weak" compatibilities with ρ, about how badly $\mathbf{V}_\Delta(\Pi)$ behaves. In particular, when ρ is reducible and $n \geq 3$, no Π compatible with P_ρ can verify $\mathbf{V}_\Delta(\Pi) \simeq \overline{L}^{\boxtimes}(\rho)$.

1 Introduction / recollection

While the mod p local Langlands correspondence is well understood for $\mathrm{GL}_2(\mathbb{Q}_p)$ (see [Bre03a] [Col10], etc.), the situation for $\mathrm{GL}_n(\mathbb{Q}_p)$ remains at the stage of groping into specific examples, especially those arising from the global picture. One of the main issues is that the theory of smooth admissible finite length $\mathrm{GL}_n(\mathbb{Q}_p)$-representations modulo p lacks both a classification of irreducibles as established by [BL94] [Bre03a] [Bre03b] for $\mathrm{GL}_2(\mathbb{Q}_p)$ and the study of extensions as in [Col10, §VII]. While partial answers can be found both in [Her11] for the classification and in [Hau16] for the extensions' study, neither address supercuspidal representations.

Let ρ be an n-dimensional representations ρ of $\mathcal{G}_{\mathbb{Q}_p}$ over an algebraic extension $\mathbb{F}$ of $\mathbb{F}_p$. In [Bre+21], five authors have formulated some conjectures about the shape of representations associated with ρ by the global picture, when the latter is generic and a good conjugate. More precisely, they define (see [Bre+21, Def. 2.4.2.7]) the notion of a smooth admissible representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ over $\mathbb{F}$ compatible with ρ. The number of Jordan-Hölder components of such representation Π are prescribed by the minimal standard parabolic P_ρ whose $\mathbb{F}$-points contain the image of ρ; this P_ρ also determines how these components fit in the classification of [Her11] (see [Bre+21, Def. 2.4.1.5]). The combinatorics of Π's subquotients is dictated by the smallest algebraic subgroup $\widetilde{P}_\rho$ containing the Levi M_{P_ρ} and whose $\mathbb{F}$-points contain the image of ρ. Finally, compatibility imposes Breuil's functor $\mathbf{V}_\xi$ to yield explicit subquotients of $\overline{L}^{\otimes}(\rho) := \bigotimes_{1\leq i<n} \Lambda^i \rho$ when applied to irreducible subquotients of Π.

Here, Breuil's functor $\mathbf{V}_\xi$ is a version of Colmez's functor[1] $\mathbf{V}$ defined in [Bre15] that produces indfinite dimensional Galois representations from smooth representations of $\mathrm{GL}_n(\mathbb{Q}_p)$. We won't need its

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[1] See [Col10, §IV.3].

precise definition. One current issue of Breuil's functor is that we do not know wether it produces finite dimensional or even nonzero objects, except for subquotients of the principal series[2] or when $n = 2$.

In [Záb18a], G. Zábrádi developed another generalisation of $\mathbf{V}$ for a general reductive group G. This functor $\mathbf{V}_\Delta$ will be the other actor of this note. Let us recall the philosophy behind Zábrádi's functor. We first investifate the intermediate functor $\mathbf{D}^\vee_\Delta$ producing multivariable (φ, Γ)-modules. In our situation, Δ is the canonical set of simple roots in the canonical root system of GL_n; it verifies $\#\Delta = n-1$. We identify $[\![1, n-1]\!]$ to Δ by sending i to the character of the torus $\mathrm{Diag}(x_1, \ldots, x_n) \mapsto x_i x_{i+1}^{-1}$ and equip Δ with the order on integers. Combining [Záb18a] and [JZ25, Th. A], the functor $\mathbf{D}^\vee_\Delta$ produces, from any finite length smooth representation Π of $\mathrm{GL}_n(\mathbb{Q}_p)$ over $\mathbb{F}$, a finite projective topological étale $T_{+,\Delta}$-module over

$$E_\Delta := \bigcup_{\mathbb{F}_q \subset \mathbb{F}} \mathbb{F}_q[\![X_\alpha \,|\, \alpha \in \Delta]\!][X_\Delta^{-1}].$$

Here $T_{+,\Delta}$ is the monoid of diagonal matrices with decreasing valuations and $X_\Delta = \prod X_\alpha$.

If Π has a central character, then $\mathbb{Q}_p^\times \mathrm{Id}_n \subset T_{+,\Delta}$ also acts via a character on $\mathbf{D}^\vee_\Delta(\Pi)$. Hence, the slightest generalisation of [Záb18b], produces an equivalence $\mathbb{V}_\Delta$ from $\mathscr{M}\mathrm{od}^{\text{ét}}(T_{+,\Delta}, E_\Delta)$ with "central character" and smooth finite dimensional representations of $(\mathcal{G}_{\mathbb{Q}_p,\Delta} \times \mathbb{Q}_p^\times)$ over $\mathbb{F}$ with "central character", i.e. $\mathbb{Q}_p^\times$ acting via homotethies. Post-composing $\mathbf{D}^\vee_\Delta$ with this equivalence and applying a suitable duality functor provides a functor $\mathbf{V}_\Delta$ to finite dimensional representations of $\mathcal{G}_{\mathbb{Q}_p}^{n-1}$ (see Definition 2.8.

Two important features are to highlight. First, ending up with multivariable (φ, Γ)-modules seems to keep more information than Breuil's functor in the same way that a representation of $\mathcal{G}_{\mathbb{Q}_p}^{n-1}$ has more information than its restriction to the diagonal $\mathcal{G}_{\mathbb{Q}_p}$. Second, Zábrádi produces in [Záb18a] a $\mathrm{GL}_n(\mathbb{Q}_p)$-equivariant sheaf on the flag variety $\mathrm{GL}_n(\mathbb{Q}_p)/B(\mathbb{Q}_p)$ from this multivariable (φ, Γ)-module associated to a $\mathrm{GL}_n(\mathbb{Q}_p)$-representation, smiliarily to what $D \boxtimes \mathbb{P}^1(\mathbb{Q}_p)$ was for $\mathrm{GL}_2(\mathbb{Q}_p)$ (see [Col10, §II.1]). From this equivariant sheaf, [JZ25] deduces finiteness results that are lacking for Breuil's functor[3].

As stated in [Bre+21, Rem. 1.2.5], the existence of Zábrádi's functor allows to formulate an improved conjecture about the Π produced by global methods from ρ. Ideally, Zábrádi's functor would take the value

$$\overline{L}^{\boxtimes}(\rho) := (\boxtimes_{1 \leq i < n}(\Lambda^i \rho))$$

on such Π and that its restriction to the diagonal embedding of $\mathcal{G}_{\mathbb{Q}_p}$ identifies with $\mathbf{V}_\xi(\Pi)$, which is isomorphic to $\overline{L}^{\otimes}(\rho)$. Such conjecture fills the following small insatisfaction: the representation $\overline{L}^{\otimes}(\rho)$ doesn't determine ρ, already in dimension 3, but $\overline{L}^{\boxtimes}(\rho)$ does.

The first aim of this note is to prove that this scenario never happens in the reducible case, even when downgrading compatibility with ρ to compatibility with $\widetilde{P}_\rho$ (see Theorem 3.5). The rough idea is that Π being compatible with $\widetilde{P}_\rho$ imposes a shortage of Jordan-Hölder components of $\mathbf{V}_\Delta(\Pi)$, in comparison to their profusion in $\overline{L}^{\boxtimes}(\rho)$. We use the same method to treat a number of irreducible cases (see Theorem 3.6).

One can argue that the condition [Bre+21, Def. 2.4.2.7 (i)] imposes heavy conditions on the image of supercuspidal by $\mathbf{V}_\Delta$. What would happen if we replaced the *loc. cit.* conditions (i)-(ii) by the single $\mathbf{V}_\xi(\Pi) = \overline{L}^{\otimes}(\rho)$? This note also gives bad behavior of $\dim \mathbf{V}_\Delta(\Pi)$ under this hypothesis for a toy example: we investigate the case of $\mathrm{GL}_3(\mathbb{Q}_p)$ and ρ being of dimension 3 and Loewy length 3 over $\overline{\mathbb{F}_p}$. For the precise result, see Proposition 5.5. Note that the bad behaviors we prove could highlight either a vanishing of Zábrádi's functor on certain supercuspidals, or a heavy lack of exactness of the same functor. Our methods don't pinpoint which failure happen[4], while always proving a slight exactness failure[5].

[2]The principal series are $\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} \chi$ for the standard Borel B and χ ranging over characters of the standard torus.

[3]They don't recover more nonvanishing results than what was known for Breuil's functor.

[4]Proving some vanishing would be a more involved result.

[5]Mainly because the image of maximal ordinary quotient of Π has a good behavior which is not reflected on the image of Π.

Notations

In this note, the field $\mathbb{F}$ is an algebraic extension of $\mathbb{F}_p$, equipped with the discrete topology so that continuous representations of locally profinite groups are smooth, i.e. have open stabilisers. All inductions are be smooth inductions.

For a finite length representation V of a group G, we denote the socle (resp. cosocle) filtration by $\mathrm{soc}^i_G V$ (resp. $\mathrm{cosoc}^i_G V$) and its graded pieces by $\mathrm{soc}_{i,G} V$ (resp. $\mathrm{cosoc}_{i,G} V$). The Loewy length is the smallest integer i such that $\mathrm{soc}^i_G = V$.

Recall the Artin map, defined by

$$\mathrm{Art}^{-1} : \mathcal{G}^{\mathrm{ab}}_{\mathbb{Q}_p} \to \widehat{\mathbb{Q}_p^\times}, \ \ \sigma \mapsto x_\sigma p^{n_\sigma}$$

where $x_\sigma \in \mathbb{Z}_p^\times$ is characterised by $\forall \zeta \in \mu_{p^\infty}, \ \sigma(\zeta) = \zeta^x$. The Weil group $\mathrm{W}_{\mathbb{Q}_p}$ verifies that $\mathrm{Art}^{-1}(\mathrm{W}^{\mathrm{ab}}_{\mathbb{Q}_p}) = \mathbb{Q}_p^\times$.

We denote by ε the smooth character of $\mathbb{Q}_p^\times$ defined by

$$\varepsilon : \mathbb{Q}_p^\times \to \mathbb{F}_p^\times, \ \ xp^n \mapsto (x \bmod p)$$

for $x \in \mathbb{Z}_p^\times$. It extends to $\widehat{\mathbb{Q}_p^\times}$.

We denote by ω the smooth character of $\mathcal{G}_{\mathbb{Q}_p}$ defined as the composition

$$\omega : \mathcal{G}_{\mathbb{Q}_p} \twoheadrightarrow \mathcal{G}^{\mathrm{ab}}_{\mathbb{Q}_p} \xrightarrow{\mathrm{Art}^{-1}} \widehat{\mathbb{Q}_p^\times} \xrightarrow{\varepsilon} \mathbb{F}_p^\times.$$

We will denote by $\mathcal{H}_{\mathbb{Q}_p} := \mathcal{G}_{\mathbb{Q}_p(\mu_{p^\infty})}$, closed normal subgroup of $\mathcal{G}_{\mathbb{Q}_p}$, with the canonical identification $\mathcal{G}_{\mathbb{Q}_p}/\mathcal{H}_{\mathbb{Q}_p} \simeq \mathbb{Z}_p^\times$ by $\sigma \mapsto x_\sigma$ as in the definition of the Artin map. We often call $\Gamma_{\mathbb{Q}_p} := \mathbb{Z}_p^\times$.

For any finite set Δ, we define $\mathcal{G}_{\mathbb{Q}_p,\Delta} := \prod_{\alpha\in\Delta} \mathcal{G}_{\mathbb{Q}_p}$ and similarly for $\mathcal{H}_{\mathbb{Q}_p,\Delta}$ and $\Gamma_{\mathbb{Q}_p,\Delta}$. We also have a canonical identification $\mathcal{G}_{\mathbb{Q}_p,\Delta}/\mathcal{H}_{\mathbb{Q}_p,\Delta} \simeq \Gamma_{\mathbb{Q}_p,\Delta}$.

For $G = \mathrm{GL}_n$, we keep the notation $\overline{L}^{\otimes}$ introduced in [Bre+21, §2.2.1]: in our situation, it is the tensor product of the antisymmetric powers of the standard algebraic representation.

For a profinite group G and a discrete field k, let $\mathrm{Rep}_k\, G$ denote the category of smooth representatons of G over k. When $G = \mathrm{GL}_n(\mathbb{Q}_p)$, it will denote the category of smooth admissible finite length representations, with center $Z(G)$ acting by homotheties.

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Contents

2 Recollection about Zábrádi's functor

Due to the intended brevity of this note, we will keep our definitions and recollections to a minimum. The first ingredient of $\mathbf{V}_\Delta$ is the multivariable Fontaine equivalence proven in [Záb18b].

Let Δ be an (abstract for now) finite set.

Definition 2.1. When $\mathbb{F}$ is finite, we define the rings

$$E_\Delta^+ := \mathbb{F}[\![X_\alpha \,|\, \alpha \in \Delta]\!] \quad \text{and} \quad E_\Delta := E_\Delta^+[X_\Delta^{-1}],$$

where $X_\Delta = \prod X_\alpha$ and equips it with an $\mathbb{F}$-linear[6] action of

$$(\Phi_{\Delta,q} \times \Gamma_{\mathbb{Q}_p,\Delta}) := \prod_{\alpha\in\Delta} \varphi_\alpha^{\mathbb{N}} \times \prod_{\alpha\in\Delta} \mathbb{Z}_p^\times$$

that verifies

$$\forall \varphi = (\varphi_\alpha^{n_\alpha})_\alpha, \forall \beta \in \Delta,\ \varphi(X_\beta) = X_\alpha^{p^{n_\alpha}}$$
$$\forall \gamma = (\gamma_\alpha)_\alpha \in \Gamma_{\mathbb{Q}_p,\Delta}, \forall \beta \in \Delta,\ \gamma(X_\beta) = (1+X_\beta)^{\gamma_\beta} - 1.$$

In general, we define E_Δ^+ and E_Δ as the colimit over finite subfields of $\mathbb{F}$ of the rings above.

Let $E := \mathbb{F}_p(\!(X)\!)$. It has the same Galois theory as $\mathbb{Q}_p(\mu_{p^\infty})$ (see for instance [Fon07, §3.1.8]) and we choose an identification $\mathcal{G}_E \simeq \mathcal{H}_{\mathbb{Q}_p}$. For each α, fix a copy $E_\alpha := \mathbb{F}_p(\!(X_\alpha)\!)$ of E, and choose an extension of the identification to $E_\alpha^{\mathrm{sep}} \simeq E^{\mathrm{sep}}$. It gives a identification of $\mathcal{G}_{E_\alpha}$ to $\mathcal{G}_E$, hence to $\mathcal{H}_{\mathbb{Q}_p}$.

Each E_α has an embedding into E_Δ and we define

$$E_\Delta^{\mathrm{sep}} := E_\alpha^{\mathrm{sep}} \otimes_{E_\alpha} \left(\dots \left(E_\beta^{\mathrm{sep}} \otimes_{E_\beta} E_\Delta \right)\right)$$

As in [Záb18b, §3.1], it has with an action of $(\Phi_{\Delta,q} \times \mathcal{G}_{\mathbb{Q}_p,\Delta})$. The α-th copy in $\mathcal{H}_{\mathbb{Q}_p,\Delta}$ acts only on the factor E_α^{sep} of the tensor product and via $\mathcal{G}_{E_\alpha} \simeq \mathcal{H}_{\mathbb{Q}_p}$.

Thanks to [Záb18b, Lem. 3.6], we have the identity

$$(E_\Delta^{\mathrm{sep}})^{\Phi_{\Delta,p}} = \mathbb{F}.$$

Thanks to [Záb18b, Prop. 3.3], we have an $(\Phi_{\Delta,p} \times \Gamma_{\mathbb{Q}_p,\Delta})$-equivariant identification of rings:

$$(E_{\Delta^{\mathrm{sep}}})^{\mathcal{H}_{\mathbb{Q}_p,\Delta}} = E_\Delta.$$

Definition 2.2 (See [Mar24, §2.2])**.** For a monoid $\mathcal{S}$ acting on a ring R, we define $\mathrm{Mod}_{\mathrm{prj}}^{\text{ét}}(\mathcal{S}, R)$ to be the category of finite projective R-module D of constant rank, equipped with a semilinear action of $\mathcal{S}$ such that

$$\forall s \in \mathcal{S},\ R \otimes_{s,R} D \xrightarrow{\mathrm{Id}\otimes s} D$$

is an isomorphism.

After this setup, we can state the multivariable cyclotomic Fontaine equivalence.

Theorem 2.3. *The following pair of functors*

$$\mathbb{D}_\Delta : \quad \mathrm{Rep}_{\mathbb{F}_p} \mathcal{G}_{\mathbb{Q}_p,\Delta} \rightleftarrows \mathrm{Mod}_{\mathrm{prj}}^{\text{ét}}(\Phi_{\Delta,p} \times \Gamma_\Delta, E_\Delta) \quad : \mathbb{V}_\Delta$$
$$V \mapsto (E_\Delta^{\mathrm{sep}} \otimes_{\mathbb{F}} V)^{\mathcal{H}_{\mathbb{Q}_p,\Delta}}$$
$$(E_\Delta^{\mathrm{sep}} \otimes_{E_\Delta} D)^{\Phi_{\Delta,p}} \leftarrow D$$

are well defined. They are quasi-inverse to one another, giving an equivalence of symmetric monoidal closed categories.

[6] As $\mathbb{F}$ keeps track of the representations' coefficients, the action of $\Phi_{\Delta,p}$ must fix it.

The right side of this equivalence is the category of multivariable cyclotomic (mod p) (φ,Γ)-modules with Δ-variables.

Remark 2.4. To pass from [Záb18b, Th. 3.15] to this theorem, one need three more ideas. The first one is that, for $\mathbb{F}$ finite, an object of $\mathrm{Rep}_{\mathbb{F}}\,\mathcal{G}_{\mathbb{Q}_p,\Delta}$ is an object of $\mathrm{Rep}_{\mathbb{F}_p}\,\mathcal{G}_{\mathbb{Q}_p,\Delta}$ with additional endomorphisms coding the external multiplication by $\mathbb{F}$. The same is true for $\mathrm{Mod}^{\text{ét}}_{\text{prj}}\left(\Phi_{\Delta,q}\times\mathcal{G}_{\mathbb{Q}_p,\Delta},E_\Delta\right)$ compared to its version over $\mathbb{F}_p$. These structure are transfered on both sides of Zábrádi's equivalence. This handles $\mathbb{F}$ being finite.

For general $\mathbb{F}$, one needs to show that $\mathrm{Rep}_{\mathbb{F}}\,\mathcal{G}_{\mathbb{Q}_p,\Delta}$ and $\mathrm{Mod}^{\text{ét}}_{\text{prj}}\left(\Phi_{\Delta,q}\times\mathcal{G}_{\mathbb{Q}_p,\Delta},E_\Delta\right)$ are colimits of the analogous categories for $\mathbb{F}'$ exhausting finite subfields of $\mathbb{F}$ along base change functors. For the Galois representations, its a general fact for a smooth representation of a compact group. For (φ,Γ)-modules, notice that $(\Phi_{\Delta,p}\times\Gamma_{\mathbb{Q}_p,\Delta})$ is topologically finitely generated and that its action on a multivariable (φ,Γ)-module is automatically continuous for some explicit topology (see [Mar25, Prop. 6.1.7]).

The last idea is that, according to [Záb18b, Prop. 2.2] [Mar25, Prop. 6.1.7], the projectivity condition is automatic.

We now prove a small useful fact that we did not see properly written anywhere.

Proposition 2.5. *Let $\Delta=\sqcup_{i\in I}\Delta_i$ be a partition of a finite set. Let $D_i\in\mathrm{Mod}^{\text{ét}}_{\text{prj}}\left(\Phi_{\Delta_i,p}\times\Gamma_{\mathbb{Q}_p,\Delta_i},E_{\Delta_i}\right)$. Then*

$$D:=\underset{i\in I,\,E_\Delta}{\otimes}\left(E_\Delta\otimes_{E_{\Delta_i}}D_i\right)$$

belongs to $\mathrm{Mod}^{\text{ét}}_{\text{prj}}\left(\Phi_{\Delta,p}\times\Gamma_{\mathbb{Q}_p,\Delta},E_\Delta\right)$ *and*

$$\mathbb{V}_\Delta(D)\simeq\boxtimes_{i\in I}\mathbb{V}_{\Delta_i}(D_i).$$

Proof. Consider $(\Phi_{\Delta_i,p}\times\Gamma_{\mathbb{Q}_p,\Delta_i})$ as a quotient of $(\Phi_{\Delta,p}\times\Gamma_{\mathbb{Q}_p,\Delta})$. This way, the map $E_{\Delta_i}\to E_\Delta$ is $(\Phi_{\Delta,p}\times\Gamma_{\mathbb{Q}_p,\Delta})$-equivariant for the action by inflation on the first term. Combine [Mar24, Prop. 3.3 1), Prop 2.15 4)] to prove that D is an étale projective module.

For the identification of $\mathbb{V}_\Delta(D)$, we first rule the case $\mathbb{F}=\mathbb{F}_p$ by adapting the strategy of [Mar25, Prop. 6.2.1]. By using [Záb18b, Lem. 3.2] for Δ and each Δ_i, then passing to the limit, we get

$$E_\Delta^{\text{sep}}\simeq\bigotimes_{i\in I,\,E_\Delta}\left(E_{\Delta_i}^{\text{sep}}\otimes_{E_{\Delta_i}}E_\Delta\right).$$

Hence

$$E_\Delta^{\text{sep}}\otimes_{E_\Delta}D\simeq\bigotimes_{i\in I,\,E_\Delta^{\text{sep}}}\left(E_\Delta^{\text{sep}}\otimes_{E_{\Delta_i}^{\text{sep}}}\left(E_{\Delta_i}^{\text{sep}}\otimes_{E_{\Delta_i}}D_i\right)\right).$$

Combining [Záb18b, Prop. 3.7] [Záb18b, Th. 3.15] we get comparison isomorphisms for each D_i, yielding

$$E_\Delta^{\text{sep}}\otimes_{E_\Delta}D\simeq\bigotimes_{i\in I,\,E_\Delta^{\text{sep}}}\left(E_\Delta^{\text{sep}}\otimes_{E_{\Delta_i}^{\text{sep}}}\left(E_{\Delta_i}^{\text{sep}}\otimes_{\mathbb{F}_p}\mathbb{V}_{\Delta_i}(D_i)\right)\right)\simeq E_\Delta^{\text{sep}}\otimes_{\mathbb{F}_p}\left(\underset{i\in I,\,\mathbb{F}_p}{\otimes}\mathbb{V}_{\Delta_i}(D_i)\right).$$

One can check that each of our isomorphisms is $(\Phi_{\Delta,p}\times\mathcal{G}_{E,\Delta})$-equivariant. Because $\boxtimes_i\mathbb{V}_{\Delta_i}(D_i)$ is an $\mathbb{F}_p$-vector space with trivial $\Phi_{\Delta,p}$-action (see [Záb18b, Lem. 3.6]), the result follows by taking $\Phi_{\Delta,p}$-invariants. □

Corollary 2.6. *For $\Delta_1\subset\Delta$ and $D_1\in\mathrm{Mod}^{\text{ét}}_{\text{prj}}\left(\Phi_{\Delta_1,p}\times\Gamma_{\mathbb{Q}_p,\Delta_1},E_{\Delta_1}\right)$,*

$$\mathbb{V}_\Delta\left(E_\Delta\otimes_{E_{\Delta_1}}D_1\right)=\mathrm{Inf}^{\mathcal{G}_{\mathbb{Q}_p,\Delta}}_{\mathcal{G}_{\mathbb{Q}_p,\Delta_1}}\mathbb{V}_{\Delta_1}(D).$$

Proof. Apply for $I=\{1,2\}$, $\Delta_2=\Delta\backslash\Delta_1$ and $D_2=E_{\Delta_2}$. □

The other ingredient of Zábrádi's functor is an analogue of Colmez's functor $\mathbf{D}^\vee$. From now on, all sets Δ will be the *standard* simple roots of $\mathrm{GL}_n(\mathbb{Q}_p)$. This is canonical simple roots of the *standard* root system obtained from the torus of diagonal matrices. It verifies $\#\Delta=n-1$.

Definition 2.7. When $\mathbb{F}$ finite, the functor $\mathbf{D}^\vee_\Delta$ is the one introduced in [Záb18a]. It takes a smooth finite length representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ over $\mathbb{F}$ with central character and produces an étale multivariable cyclotomic (φ, Γ)-module with Δ-variables.

In general, as any smooth representation Π of $\mathrm{GL}_n(\mathbb{Q}_p)$ over $\mathbb{F}$ is defined over a finite field[7], we can define

$$\mathbf{D}^\vee_\Delta(\Pi) := \varprojlim_{\substack{\Pi'/\mathbb{F}_q \text{ s.t.} \\ \Pi = \mathbb{F}\otimes_{\mathbb{F}_q}\Pi'}} \mathbb{F} \otimes_{\mathbb{F}_q} \mathbf{D}^\vee_\Delta(\Pi')$$

with the above definition for $\mathbf{D}^\vee_\Delta(\Pi')$. One verifies that the category is filtered and all the transition maps in the limit are isomorphisms.

We finally combine the multivariable Fontaine equivalence and Zábrádi's functor.

Definition 2.8. Let $\delta^{\boxtimes}_\Delta$ be the character $\boxtimes_{i\in\Delta}\omega^{\sum_{j=n-i}^{n-1} j}$ of $\mathcal{G}^{n-1}_{\mathbb{Q}_p}$, whose restriction to the diagonal embedding is $\delta_{\mathrm{GL_n}}$ from [Bre+21, p. 23]. We define $\mathbb{V}^*_\Delta := \underline{\mathrm{Hom}}(\mathbb{V}_\Delta, \delta^{\boxtimes}_\Delta)$.

Definition 2.9. We define $\mathbf{V}_\Delta := \mathbb{V}^*_\Delta \circ \mathbf{D}^\vee_\Delta$. It's a contravariant functor from the category $\mathrm{Rep}_{\mathbb{F}}\,\mathrm{GL}_n(\mathbb{Q}_p)$ to $\mathrm{Rep}_{\mathbb{F}}\,\mathcal{G}_{\mathbb{Q}_p,\Delta}$.

We define $\mathbf{V}_{\Delta,\xi} := \mathrm{Res}^{\mathcal{G}_{\mathbb{Q}_p,\Delta}}_{\mathcal{G}_{\mathbb{Q}_p}} \circ \mathbf{V}_\Delta$ where the restriction is along the diagonal embedding.

Recall three important theorems about Zábrádi's functor.

Theorem 2.10. *[See [Záb18a, Th. C, Th. G]] The functor $\mathbf{D}^\vee_\Delta$ is right exact. It is exact on the subcategory SP of representations whose Jordan-Hölder factors are subquotients of principal series. It is fully faithful on the subcategory SP^0 of representations whose Jordan-Hölder factors are irreducible principal series.*

Dual results holds for $\mathbf{V}_\Delta$ as both $\mathbb{V}_\Delta$ and $\underline{\mathrm{Hom}}$ are exact.

Theorem 2.11. *[See [JZ25, Th. C]] If Π is irreducible smooth, then $\mathbf{V}_\Delta(\Pi)$ is either zero or irreducible.*

Theorem 2.12. *[See [Záb18a, Prop. 3.2, Th. 3.5]] Let P be a standard parabolic of GL_n. Write its Levi subgroup as the diagonal $\prod_{1\le t\le s}\mathrm{GL}_{m_s}$. The intersection of the standard root system of GL_{m_t} with Δ give the standard simple roots Δ_t of GL_{m_t}.*

For all t, let Π_t be a smooth admissible representation of $\mathrm{GL}_{m_t}(\mathbb{Q}_p)$ over $\mathbb{F}$. We see $\boxtimes_t \Pi_t$ as a representation of $P^-(\mathbb{Q}_p)$ by inflation. Then,

$$\mathbf{D}^\vee_\Delta\Big(\mathrm{Ind}^{\mathrm{GL}_n(\mathbb{Q}_p)}_{P^-(\mathbb{Q}_p)}(\boxtimes_t \Pi_t)\Big) \simeq \bigotimes_{1\le t\le s} \big(E_\Delta \otimes_{E_{\Delta_t}} \mathbf{D}^\vee_{\Delta_t}(\Pi_t)\big).$$

Definition 2.13. We define a canonical surjection $E_\Delta \twoheadrightarrow E$ as the quotient by the ideal $(X_\alpha - X_\beta \,|\, \alpha,\beta\in\Delta)$, i.e. the identification of all variables to X. It is $(\varphi^{\mathbb{N}}\times\Gamma)$-equivariant for its action on E_Δ via diagonal embedding of $(\varphi^{\mathbb{N}}\times\Gamma)$ into $(\Phi_{\Delta,p}\times\Gamma_{\mathbb{Q}_p,\Delta})$.

Using the identifications $E^{\mathrm{sep}}_\alpha \simeq E^{\mathrm{sep}}$ we deduce a map

$$\bigotimes_{\alpha\in\Delta,\, E_\Delta} (E^{\mathrm{sep}}_\alpha \otimes_{E_\alpha} E_\Delta) \to E^{\mathrm{sep}}$$

i.e. a surjection $E^{\mathrm{sep}}_\Delta \twoheadrightarrow E^{\mathrm{sep}}$. It is $(\varphi^{\mathbb{N}}\times\mathcal{G}_{\mathbb{Q}_p})$-equivariant for its action on E^{sep}_Δ via diagonal embedding of $(\varphi^{\mathbb{N}}\times\mathcal{G}_{\mathbb{Q}_p})$ into $(\Phi_{\Delta,p}\times\mathcal{G}_{\mathbb{Q}_p,\Delta})$.

Lemma 2.14. *There is a natural isomorphism of functors from $\mathrm{Mod}^{\text{ét}}_{\mathrm{prj}}\left(\Phi_{\Delta,p}\times\Gamma_{\mathbb{Q}_p,\Delta}, E_\Delta\right)$ to $\mathrm{Rep}_{\mathbb{F}}\,\mathcal{G}_{\mathbb{Q}_p}$*

$$\mathrm{Res}^{\mathcal{G}_{\mathbb{Q}_p,\Delta}}_{\mathcal{G}_{\mathbb{Q}_p}} \circ \mathbb{V}_\Delta \Rightarrow \mathbb{V}\left(E\otimes_{E_\Delta} -\right).$$

Here, the restriction is along the diagonal embedding and the tensor product is correctly defined using [Mar24, Prop. 3.3] for the $(\varphi^{\mathbb{N}}\times\Gamma)$-equivariant canonical map.

[7]Use that $\mathrm{GL}_n(\mathbb{Q}_p)$ is generated by $\mathrm{GL}_n(\mathbb{Z}_p)$ and a finite number of elements.

Proof. Factorising $E_\Delta \twoheadrightarrow E \hookrightarrow E^{\mathrm{sep}}$ through E_Δ^{sep}, we obtain a natural map

$$(E_\Delta^{\mathrm{sep}} \otimes_{E_\Delta} D)^{\Phi_{\Delta,p}} \to \left(E^{\mathrm{sep}} \otimes_{E_\Delta^{\mathrm{sep}}} (E_\Delta^{\mathrm{sep}} \otimes_{E_\Delta} D)\right)^{\varphi=\mathrm{Id}}.$$

It is a map of $\mathcal{G}_{\mathbb{Q}_p}$-representations, hence defines a natural transformation as required.

To verify that it is a natural isomorphism, we can forget the action of $\mathcal{G}_{\mathbb{Q}_p}$. The isomorphism of comparison, proved together with multivariable cyclotomic Fontaine equivalence, implies that

$$E_\Delta^{\mathrm{sep}} \otimes_{E_\Delta} D \simeq (E_\Delta^{\mathrm{sep}})^{\oplus \mathrm{rk} D}$$

in $\mathrm{Mod}_{\mathrm{prj}}^{\text{ét}}(\Phi_{\Delta,p}, E_\Delta^{\mathrm{sep}})$. Hence, proving that the map constructed above is an isomorphism boils down to the equalities $(E_\Delta^{\mathrm{sep}})^{\Phi_{\Delta,p}} = (E^{\mathrm{sep}})^{\varphi=\mathrm{Id}} = \mathbb{F}$ and the $\mathbb{F}$-linearity of all the maps in Definition 2.13. □

Corollary 2.15. *For any smooth representation* Π *of* $\mathrm{GL}_n(\mathbb{Q}_p)$*, we have a natural injection* $\mathbf{V}_{\Delta,\xi}(\Pi) \hookrightarrow \mathbf{V}_\xi(\Pi)$.

Proof. Using[8] [Záb18a, Rem. 2 p. 13], there is a natural surjection

$$\mathbf{D}_\xi^\vee(\Pi) \twoheadrightarrow E \otimes_{E_\Delta} \mathbf{D}_\Delta^\vee(\Pi).$$

Let χ_{GL_n} be the 1-dimensional (φ,Γ)-module such that $\mathbb{V}(\chi_{\mathrm{GL}_n}) = \delta_{\mathrm{GL}_n}$ (the latter is defined in [Bre+21, p. 23]). It coincides with base change along the canonical $E_\Delta \to E$ of the multivariable (φ,Γ)-module $\chi_\Delta^\boxtimes$ such that $\mathbb{V}_\Delta(\chi_\Delta^\boxtimes) = \delta_\Delta^\boxtimes$. We pass to duals and twist by χ_{GL_n}, obtaining

$$\underline{\mathrm{Hom}}(E \otimes_{E_\Delta} \mathbf{D}_\Delta^\vee(\Pi), \chi_{\mathrm{GL}_n}) \hookrightarrow \underline{\mathrm{Hom}}(\mathbf{D}_\xi^\vee, \chi_{\mathrm{GL}_n}).$$

By [Mar24, Prop. 3.3 3)], internal Hom and base change commute, which identifies the left term to $E \otimes_{E_\Delta} \underline{\mathrm{Hom}}(\mathbf{D}_\Delta^\vee(\Pi), \chi_\Delta^\boxtimes) \simeq E \otimes_{E_\Delta} \mathbf{V}_\Delta(\Pi)$. We apply Fontaine's functor $\mathbb{V}$ to the previous injection, which still produces an injection thanks to $\mathbb{V}$'s exactness. The left term now identifies with $\mathbf{V}_{\Delta,\xi}(\Pi)$ by Lemma 2.14. The right term now identifies with $\mathbf{V}_\xi(\Pi)$ thanks to the commutation of $\mathbb{V}$ to internal Hom. □

3 Imprecise study of $\overline{L}^\boxtimes(\rho)$ and first failure

In this section, we highlight a first failure of $\mathbf{V}_\Delta$ relying on two ingredients: [JZ25, Th. C] and counting Jordan-Hölder components.

In this section, we fix an n-dimensional object ρ of $\mathrm{Rep}_{\mathbb{F}}\,\mathcal{G}_{\mathbb{Q}_p}$. We fix a composition serie for ρ and write the successive irreducible pieces $(\rho_l)_{1\le l\le k}$ of respective dimensions $\underline{n} = (n_l)$.

Definition 3.1. For such a tuple $\underline{n}$ and $i \geq 1$, define the set of cuts

$$\mathrm{Cut}_{i,\underline{n}} := \left\{(j_l) \in \mathbb{N}_{\geq 1}^{[\![1,k]\!]} \,\middle|\, {\textstyle\sum j_l = i \atop \forall l,\ 1\le i_l\le n_l}\right\}.$$

Lemma 3.2. *Let* $1 \leq i < n$*. The representation* $\Lambda^i\rho$ *has more that* $\#\mathrm{Cut}_{i,\underline{n}}$ *Jordan-Hölder components.*

Proof. As our composition serie for ρ produces one for $\Lambda^i\rho$, the claim is equivalent to its analogue for $\rho^{\mathrm{ss}} = \oplus_l \rho_l$. In this case, we have an isomorphism

$$\bigoplus_{(j_l)\in \mathrm{Cut}_{i,\underline{n}}} \bigotimes_{1\le l\le k} \left(\Lambda^{j_l}\rho_l\right) \to \Lambda^i \rho^{\mathrm{ss}}$$

with only nonzero terms in the sum. It concludes. □

[8] And the finiteness proved in [JZ25].

Corollary 3.3. *The representation $\overline{L}^{\boxtimes}(\rho)$ has more that $\prod_{1\le i<n} \#\mathrm{Cut}_{i,\underline{n}}$ Jordan-Hölder components.*

Proof. Combine the previous lemma with the following general fact. If V and W are representations of G and H with two normal series $(V_a)_{a\le A}$ and $(W_b)_{b\le B}$, then $(V_{a-1}\boxtimes W + V_a\boxtimes W_b)_{a,b\le A,B}$ with lexicographic order is a normal serie for $V\boxtimes W$. □

Until the end of the section, suppose that ρ is generic and a good conjugate[9] in the sense of [Bre+21, §2.3.2]. With this assumption P_ρ is the standard parabolic whose Levi is $M_{P_\rho} = \mathrm{GL}_{n_1}\times\ldots\times\mathrm{GL}_{n_k}$.

Lemma 3.4. *The number of isotypic components of $\overline{L}^{\otimes}_{|Z_{M_{P_\rho}}}$ is smaller than $\prod_{1\le i<n}\#\mathrm{Cut}_{i,\underline{n}}$, strictly if ρ is not irreducible and $n\ge 3$.*

Proof. Call $(e_r)_{1\le r\le n}$ the canonical basis of the standard representation. The family $(\otimes_i(\wedge_{r\in I_i}e_r))_{(I_i)}$ for (I_i) ranging over in $\prod_{1\le i<n}\{I\subset [\![1,n]\!]\,|\,\#I=i\}$ forms a basis of $Z_{M_{P_\rho}}$-eigenvectors of $\overline{L}^{\otimes}$. To each eigenvector corresponds an algebraic character of $Z_{M_\rho}\simeq\mathbb{G}_m^k$ i.e. a tuple of integers, which are positive for the standard representation. The isotypic components are hence parametrised by

$$J_{\underline{n}} := \left\{(\mu_l)\in\mathbb{N}^{[\![1,k]\!]}\ \middle|\ \substack{\exists (I_i)_{1\le i<n} \text{ subsets of } [\![1,n]\!]\\ \text{s.t. } \forall i,\, \#I_i=i\\ \text{and } \forall l,\, \sum_i \#(I_i\cap[\![\sum_{l'<l}n_{l'}+1,\sum_{l'\le l}n_{l'}]\!])=\mu_l}\right\}.$$

We will say that (I_i) is a witness of $(\mu_l)\in J_{\underline{n}}$ if they verify the condition defining $J_{\underline{n}}$.

Consider the map

$$\prod_{1\le i<n}\mathrm{Cut}_{i,\underline{n}}\to J_{\underline{n}},\ \ (j_{i,l})_{i,l}\mapsto\left(\sum_i j_{i,l}\right)_l.$$

It is well defined because $(I_i := \sqcup_l[\![\sum_{l'<l}n_{l'}+1,\sum_{l'<l}n_{l'}+j_{i,l})$ is a witness of the image of $(j_{i,l})$. It is also surjective, cause any (μ_l) with witness (I_i) is the image of $(\#(I_i\cap[\![\sum_{l'<l}n_{l'}+1,\sum_{l'\le l}n_{l'}]\!]))_{i,l}$.

This finishes to prove that the number of isotypic components is small than the cardinal of tuples of cuts in $\prod\mathrm{Cut}_{i,\underline{n}}$. Proving that it is strictly smaller amounts to finding a default of injectivity of the map above when ρ is reducible (i.e. $k\ge 2$) and $n\ge 3$.

If $k\ge 3$, we get inspiration from the example of $n=3$ and $k=3$, i.e. $P_\rho = B\subset\mathrm{GL}_3$. For this example, $e_1\otimes(e_2\wedge e_3)$ and $e_2\otimes(e_1\wedge e_3)$ are two eigenvectors for the same character $\mathrm{Diag}(x,y,z)\mapsto xyz$ of the torus, but they correspond to two pairs of cuts $((1,0,0),(0,1,1))$ and $((0,1,0),(1,0,1))$ that are witnesses of (the tuple associated with) the character. We mimicry this in the general situation. Take any family $(I_i)_{3\le i<n}$ and look at (μ_l) corresponding to $(\{1\},\{2,3\},I_3,\ldots,I_{n-1})$. Then $(\{2\},\{1,3\},I_3,\ldots,I_{n-1})$ also witnesses the belonging of (μ_l) to $J_{\underline{n}}$. These two witnesses give two elements in the preimage of (μ_l).

If $k=2$, there exists an l_0 such that $n_{l_0}\ge 2$. We get inspiration from the example of $n=3$ and $(n_1,n_2)=(1,2)$, i.e. P_ρ is the parabolic associated to $\mathbb{G}_m\times\mathrm{GL}_2$. In this situation, the same two vectors as before are eigenvectors for the character $\mathrm{diag}(x,y,y)=xy^2$ of $Z_{M_{P_\rho}}$ but produce different witnesses of (the tuple associated with) the character. We mimicry this in the general situation fixing l_0 as above and $l_1\ne l_0$. Take any family $(I_i)_{3\le i<n}$ and look at (μ_l) corresponding to $(\{\sum_{l<l_0}n_l+1\},\{\sum_{l<l_1}n_l+1,\sum_{l<l_0}n_l+2\},I_3,\ldots,I_{n-1})$. In this situation, the family $(\{\sum_{l<l_1}n_l+1\},\{\sum_{l<l_0}n_l+1,\sum_{l<l_0}n_l+2\},I_3,\ldots,I_{n-1})$ also witnesses the belonging of (μ_l) to $J_{\underline{n}}$. Same conclusion. □

Theorem 3.5. *Let $n\ge 3$ and ρ be a n-dimensional reducible object of $\mathrm{Rep}_{\mathbb{F}}\,\mathcal{G}_{\mathbb{Q}_p}$, which is generic and a good conjugate. Then any Π in $\mathrm{Rep}_{\mathbb{F}}\,\mathrm{GL}_n(\mathbb{Q}_p)$ that is compatible with $\widetilde{P}_\rho$ verifies*

$$\mathbf{V}_\Delta(\Pi)\not\simeq\overline{L}^{\boxtimes}(\rho).$$

[9] This is very mild asumption: genericity takes away multiplicities and the pathological extensions of characters already appearing for $\mathrm{GL}_2(\mathbb{Q}_p)$ (see [Col10, VII.4 Atomes de longueur 3]). Under this hypothesis, [Bre+21, §2.3.2] shows that good conjugates exist and all have similar notions of compatibility.

Proof. We first investigate the image of a Jordan-Hölder component of Π by $\mathbf{V}_\Delta$. For each one of these, the conditions (i) and (iv) of [Bre+21, Def. 2.4.1.5] tell that there exist a standard parabolic Q with Levi $M_Q = \mathrm{GL}_{m_1} \times \ldots \times \mathrm{GL}_{m_s}$ and supercuspidal representations[10] Π_t of $\mathrm{GL}_{m_t}(\mathbb{Q}_p)$ such that the component is isomorphic to $\mathrm{Ind}_{Q^-(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)}(\boxtimes_t \Pi_t)$. As usual $\boxtimes_t \Pi_t$ is seen as a $Q^-(\mathbb{Q}_p)$-representation via inflation from M_Q. Thanks to Theorem 2.12, we have

$$\mathbf{D}_\Delta^\vee\Big(\mathrm{Ind}_{Q^-(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)}(\boxtimes_t \Pi_t)\Big) \simeq \bigotimes_{1\leq t\leq s} \big(E_\Delta \otimes_{E_{\Delta_t}} \mathbf{D}_{\Delta_t}^\vee(\Pi_t)\big)$$

where Δ_t is set standard set of simple roots of GL_{m_t}. We then use Proposition 2.5 to see that

$$(\mathbb{V}_\Delta \circ \mathbf{D}_\Delta^\vee)\Big(\mathrm{Ind}_{Q^-(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)}(\boxtimes_t \Pi_t)\Big) \simeq \underset{1\leq t\leq s}{\boxtimes} (\mathbb{V}_{\Delta_t} \circ \mathbf{D}_{\Delta_t}^\vee)(\Pi_t).$$

In our situation, tensor product and dual commutes; the representation $\mathbf{V}_\Delta(\mathrm{Ind}_{Q^-(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)}(\boxtimes \Pi_t))$ is the inflation $\mathrm{Inf}_{\prod_t \mathcal{G}_{\mathbb{Q}_p,\Delta_t}}^{\mathcal{G}_{\mathbb{Q}_p,\Delta}}\big(\boxtimes_t \mathbf{V}_{\Delta_t}(\Pi_t)\big)$ along the projection $\mathcal{G}_{\mathbb{Q}_p,\Delta} \to \prod \mathcal{G}_{\mathbb{Q}_p,\Delta_t}$, twisted by a character. Each $\mathbf{V}_{\Delta_t}(\Pi_t)$ is zero or irreducible thanks to [JZ25, Th. C]. Hence, so is the inflation of their boxproduct. We showed the image by $\mathbf{V}_\Delta$ of each Jordan-Hölder factor of Π is either zero or irreducible.

Thanks to [Bre+21, Lem. 2.4.1.1 (i)], the number of Jordan-Hölder components of Π is precisely the number of isotypic components of $\overline{L}^{\otimes}_{|Z_{M_{P_\rho}}}$ i.e. $J_{\underline{n}}$. Using right exactness of $\mathbf{V}_\Delta$, the number of Jordan-Hölder components of $\mathbf{V}_\Delta(\Pi)$ is less than $\#J_{\underline{n}}$. Thanks to Corollay 3.3 and Lemma 3.4, this is less than the number of irreducible components of $\overline{L}^{\boxtimes}(\rho)$. The two Galois representations are not isomorphic. □

The same counting of Jordan-Hölder components can work for irreducible representations.

Theorem 3.6. *There exist infinitely many integers n such that there is an n-dimensional irreducible $\rho \in \mathrm{Rep}_{\mathbb{F}}\, \mathcal{G}_{\mathbb{Q}_p}$, with no supercuspidal $\Pi \in \mathrm{Rep}_{\mathbb{F}}\, \mathrm{GL}_n(\mathbb{Q}_p)$ verifying $\mathbf{V}_\Delta(\Pi) \simeq \overline{L}^{\boxtimes}(\rho)$.*

Proof. By Theorem 2.11, any $\mathbf{V}_\Delta(\Pi)$ for Π supercuspidal is either zero or irreducible. As soon as $\Lambda^i\rho$ is reducible for some $i \geq 2$, then $\overline{L}^{\boxtimes}(\rho)$ is reducible; we will construct a family of representations ρ with unbounded dimensions for which $\Lambda^2\rho$ is reducible.

Recall from local class field theory that

$$\mathcal{G}_{\mathbb{Q}_p}/\mathcal{I}_{\mathbb{Q}_p}^{\mathrm{wild}} \simeq \left(\varprojlim_{r\geq 1,\ \mathrm{norm\ map}} \mathbb{F}_{p^r}^\times\right) \rtimes \varphi^{\widehat{\mathbb{Z}}}.$$

Take any prime ℓ with $\ell \nmid p(p^2-1)(p^3-1)$ and $p \nmid (\ell - 1)$. They are infinitely many such thanks to Dirichlet prime number theorem. As $\ell | (p^r - 1)$ for some r, we can identify $\mathbb{Z}/\ell\mathbb{Z}$ to a quotient of $\varprojlim \mathbb{F}_{p^r}^\times$ by a characteristic subgroup H. The conjugaison action of $\varphi^{\widehat{\mathbb{Z}}}$ on this quotient identifies to the multiplication by some element in $(\mathbb{Z}/\ell\mathbb{Z})^\times$. Write the kernel of this action as $\varphi^{m\widehat{\mathbb{Z}}}$. Consider the quotient Q_ℓ of $\mathcal{G}_{\mathbb{Q}_p}/\mathcal{I}_{\mathbb{Q}_p}^{\mathrm{wild}}$ by $H \rtimes \varphi^{m\widehat{\mathbb{Z}}}$ that we identify to $(\mathbb{Z}/\ell\mathbb{Z}) \rtimes K_\ell$ for some $K_\ell \lhd (\mathbb{Z}/\ell\mathbb{Z})^\times$. Remark that $\#K_\ell$ is the order of p in $(\mathbb{Z}/\ell\mathbb{Z})^\times$. For any d, the primes ℓ stop dividing $p^d - 1$ at some point, hence $\#K_\ell \xrightarrow{\ell\to+\infty} +\infty$.

We now look more precisely at the irreducible representations of Q_ℓ over $\overline{\mathbb{F}_p}$. The conjugacy classes of Q_ℓ are the sets $\{\mathrm{Id}\}$, $aK_\ell \times \{1\}$ for $a \in (\mathbb{Z}/\ell\mathbb{Z})\backslash\{0\}$ and $(\mathbb{Z}/\ell\mathbb{Z}) \times \{x\}$ for $x \in K_\ell\backslash\{1\}$: there are $\#K_\ell + (l-1)/\#K_\ell$ of them. We also have $\#K_\ell$ irreducible non isomorphic 1-dimensional representations, obtained by inflation from K_ℓ. Our hypotheses on ℓ imply that $p \nmid \#Q_\ell$; Frobenius theorem gives

$$\#K_\ell + (l-1)m^2/(\#K_\ell) \geq \sum_{\substack{[V]\text{ isomorphism classes of}\\ \text{irreducible representations of } Q_\ell \text{ over } \overline{\mathbb{F}_p}}} (\dim V)^2 = \#Q_\ell = \ell(\#K_\ell)$$

[10] This is equivalent to $\boxtimes_t \Pi_t$ being supercuspidal.

where m is the maximum dimension of such irreducible representation. In particular, we have $m \geq \#K_\ell$. Pick one irreducible of dimension m. The (finite) sum of its distinct conjugates by $\mathcal{G}_{\mathbb{F}}$ descends to an irreducible representation over $\mathbb{F}$. It must be of maximal dimension among such irreducibles and this dimension is greater than $\#K_\ell$. We fix such a representation σ_ℓ. Because $\ell \nmid (p^2-1)(p^3-1)$, we have $\#K_\ell \geq 4$. As $\dim \Lambda^2 \sigma_\ell = \dim \sigma_\ell (\dim \sigma_\ell - 1)/2 > \dim \sigma_l$, the representation $\bigwedge^2 \sigma_\ell$ can't be irreducible.

Adding everything, the inflations $\mathrm{Inf}_{Q_\ell}^{\mathcal{G}_{\mathbb{Q}_p}} \sigma_\ell$ form an family of irreducible representations with unbounded dimensions and reducible Λ^2. □

4 Setup for the GL_3-toy case

Our toy example for studying a "weak compatibility" hypothesis is a 3-dimensional representation ρ of $\mathcal{G}_{\mathbb{Q}_p}$ of length 3, which we suppose to be generic and good conjugate. Here, it means that ρ can be written

$$\begin{pmatrix} \chi_1 & \delta_a & \epsilon \\ & \chi_2 & \delta_b \\ & & \chi_3 \end{pmatrix}$$

with all $\chi_i \chi_j^{-1} \notin \{1, \omega^{\pm 1}\}$.

The first thing we do is analyse the shape of $\overline{L}^{\boxtimes}(\rho)$ and $\overline{L}^{\otimes}(\rho)$. Recall a slight variant of the notion of module diagram introduced in [Alp80].

Definition 4.1 (Variant of [Alp80, §3-4])**.** A module diagram is the finite directed graph $\mathcal{D}$ obtained from a partially ordered finite set X by taking X as set of vertices and adjoining one edge from x to y if $x < y$ and if there does not exist $x < z < y$.

Let V be a finite length representation of a group G over a field k. A labelled module diagram $\mathcal{D}$ for V is a pair $(\mathcal{D}, \delta)$ where $\mathcal{D}$ is a module diagram with $\mathrm{length}\, V$ vertices and δ is a function from initial subgraphs[11] of $\mathcal{D}$ to subrepresentations of V which verifies $\delta(\mathcal{D}) = V$ and preserves joints, intersections and radicals[12].

Let $(\mathcal{D}, \delta)$ be a labelled module diagram for V. For each vertex x, call $\mathcal{D}_{\leq x} := \{y \in \mathcal{D} \,|\, y \leq x\}$. Then, [Alp80, Th. 3] proves that $x \mapsto \delta(\mathcal{D}_{\leq x})/\delta(\mathrm{rad}\, \mathcal{D}_{\leq x})$ establishes a bijection between the vertices of $\mathcal{D}$ and the set $\mathrm{JH}(V)$ of Jordan-Hölder components of V counted with multiplicity. We will often synthetise a labelled module diagram by writing only the directed graph with vertices labelled by $\mathrm{JH}(V)$ (which contains slightly less information).

Remark 4.2. For any total ordering $\preceq$ on $\mathrm{JH}(V)$ extending the graph order, evaluating δ at $\mathcal{D}_{\preceq x} := \{y \preceq x\}$ provides a composition serie for V with gradings $\mathrm{JH}(V)$ in the order $\preceq$.

There can be several labelled module diagrams for V. As basic examples, if V is a direct sum of irreducibles $V = \oplus_{i \leq n} V_i$, any partial order $\leq$ on $\{V_i\}$, gives a labelled module diagram with

$$\delta(\mathcal{D}') = \bigoplus_{V_j \text{ vertex of } \mathcal{D}'} V_i.$$

Worse, if $V = W^{\oplus 2}$ with W irreducible, any couple of embeddings $\iota_1, \iota_2 : W \hookrightarrow V$ with distincts images gives a labelled module diagram where $\mathcal{D}$ has vertices $\{1, 2\}$, no edge, and where

$$\delta(\mathcal{D}') = \sum_{j \text{ vertex of } \mathcal{D}'} \mathrm{Im}(\iota_j).$$

When V is multiplicity free, there exists an initial labelled module diagram for V. Its directed graph is constructed in [Alp80, §2], and has underlying partial order on Jordan-Hölder factors defined by $W_1 \leq W_2$ iff $V(W_1) \subseteq V(W_2)$ where $V(W)$ is the unique subrepresentation with cosocle W.

[11] I.e. $\mathcal{D}' \subset \mathcal{D}$ such that $x \to y$ and $y \in \mathcal{D}'$ imply $x \in \mathcal{D}'$.

[12] The radical of a graph is the intersection of maximal strict initial subgraphs.

Lemma 4.3. *The representation $\Lambda^2\rho$ has length 3 and a labelled module diagram looking like*

$$\chi_1\chi_2 \longrightarrow \chi_1\chi_3 \longrightarrow \chi_2\chi_3$$

Its socle (resp. its cosocle) is of same dimension as the cosocle (resp. the socle) of ρ.

Proof. In the basis $\{e_1 \wedge e_2,\ e_1 \wedge e_3,\ e_2 \wedge e_3\}$ the matrix representation is

$$\begin{pmatrix} \chi_1\chi_2 & \chi_1\delta_b & \delta_a\delta_b - \epsilon\chi_2 \\ & \chi_1\chi_3 & \delta_a\chi_3 \\ & & \chi_2\chi_3 \end{pmatrix}.$$

Let's give one argument for socle/cosocle analysis, the other being similar. The subrepresentation $\mathrm{Vect}(e_1 \wedge e_2,\ e_1 \wedge e_3)$ is semisimple iff δ_b is trivial, hence iff ρ/χ_1 is semisimple. $\square$

Proposition 4.4. *The representation $\overline{L}^{\boxtimes}(\rho)$ has a labelled module diagram looking like*

$$\begin{tikzcd}
 & & \chi_1 \boxtimes (\chi_2\chi_3) \arrow[dr] & & \\
 & \chi_1 \boxtimes (\chi_1\chi_3) \arrow[ur] \arrow[dr] & & \chi_2 \boxtimes (\chi_2\chi_3) \arrow[dr] & \\
\chi_1 \boxtimes (\chi_1\chi_2) \arrow[ur] \arrow[dr] & & \chi_2 \boxtimes (\chi_1\chi_3) \arrow[ur] \arrow[dr] & & \chi_3 \boxtimes (\chi_2\chi_3) \\
 & \chi_2 \boxtimes (\chi_1\chi_2) \arrow[ur] \arrow[dr] & & \chi_3 \boxtimes (\chi_1\chi_3) \arrow[ur] & \\
 & & \chi_3 \boxtimes (\chi_1\chi_2) \arrow[ur] & &
\end{tikzcd}$$

When ρ has Loewy length 3 (resp. 2, resp. is semisimple), $\overline{L}^{\boxtimes}(\rho)$ has Loewy length 5 (resp. 3, resp. 1) with dimensions of the graded pieces $(1, 2, 3, 2, 1)$ (resp. $(2, 5, 2)$, resp. 9).

Proof. Use the convolution formula for boxproducts of smooth representations of finite dimension. One could also write down explictely the exterior tensor product of matrices. $\square$

We write $\overline{L}^{\boxtimes}(\rho)^{\mathrm{ord}}$ for the subrepresentation of $\overline{L}^{\boxtimes}(\rho)$ having $(\chi_1 \boxtimes \chi_1\chi_2)$, $(\chi_1 \boxtimes \chi_1\chi_3)$ and $(\chi_2 \boxtimes \chi_1\chi_2)$ as Jordan-Hölder factors. In case ρ is of Loewy length 3 (equivalently when it is maximally non split), it's $\mathrm{soc}_{2,(\mathcal{G}_{\mathbb{Q}_p}\times\mathcal{G}_{\mathbb{Q}_p})}\overline{L}^{\boxtimes}(\rho)$. We exchange the symbol $\boxtimes$ for an $\otimes$ to denote their restriction to the diagonal embedding of $\mathcal{G}_{\mathbb{Q}_p}$.

Remark 4.5. The representation $\overline{L}^{\otimes}(\rho)^{\mathrm{ord}}$ coincide with $(\overline{L}^{\otimes}_{|B})^{\mathrm{ord}} \circ \rho$ where the first is defined in [BH15, §2.5]. See also [Bre15] where this composition appears.

Corollary 4.6. *If $p \neq 2$, the $\mathcal{G}_{\mathbb{Q}_p}$-representation $\overline{L}^{\otimes}(\rho)/\overline{L}^{\otimes}(\rho)^{\mathrm{ord}}$ has a labelled module diagram looking like*

$$\begin{tikzcd}
\det\rho \arrow[r] & \chi_2^2\chi_3 \arrow[dr] & \\
\det\rho \arrow[rr] & & \chi_2\chi_3^2 \\
\det\rho \arrow[r] & \chi_1\chi_3^2 \arrow[ur] &
\end{tikzcd}$$

when ρ is of Loewy length 3, the columns give the socle filtration.

Proof. In the basis $\{e_1 \otimes (e_2 \wedge e_3), e_2 \otimes (e_1 \wedge e_3), e_3 \otimes (e_1 \wedge e_2), e_2 \otimes (e_2 \wedge e_3), e_3 \otimes (e_1 \wedge e_3), e_3 \otimes (e_2 \wedge e_3)\}$, the matrix looks like

$$\begin{pmatrix} \det\rho & 0 & 0 & \chi_2\chi_3\delta_a & 0 & * \\ & \det\rho & 0 & \chi_2\chi_3\delta_a & \chi_1\chi_3\delta_b & * \\ & & \det\rho & 0 & \chi_1\chi_3\delta_b & * \\ & & & \chi_2^2\chi_3 & 0 & \chi_2\chi_3\delta_b \\ & & & & \chi_1\chi_3^2 & \chi_3^2\delta_a \\ & & & & & \chi_2\chi_3^2 \end{pmatrix}$$

We apply the base change by the matrix $\left(\begin{smallmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & \mathrm{Id}_2 \end{smallmatrix}\right)$, which invertible as soon as $p \neq 2$. We obtain

$$\begin{pmatrix} \det\rho & 0 & 0 & \chi_2\chi_3\delta_a & 0 & * \\ & \det\rho & 0 & 0 & 0 & * \\ & & \det\rho & 0 & \chi_1\chi_3\delta_b & * \\ & & & \chi_2^2\chi_3 & 0 & \chi_2\chi_3\delta_b \\ & & & & \chi_1\chi_3^2 & \chi_3^2\delta_a \\ & & & & & \chi_2\chi_3^2 \end{pmatrix}$$

which gives the announced results.

□

Remark 4.7. Be careful that socle filtration and restriction cto the diagonal embedding do not commute: for instance, there's always a copy of $\det\rho$ in $\mathrm{soc}^2_{\mathcal{G}_{\mathbb{Q}_p}}(\overline{L}^{\otimes}(\rho))$.

For ρ of length 3, the representation predicted by [Bre+21, §2.4.3 Ex. 2] has labelled module diagram looking like

$$\begin{tikzcd} & \mathrm{PS}_{\chi_2,\chi_1,\chi_3} \arrow[dr] & & \mathrm{PS}_{\chi_2,\chi_3,\chi_1} \arrow[dr] & \\ \mathrm{PS}_{\chi_1,\chi_2,\chi_3} \arrow[ur] \arrow[dr] & & \mathrm{SS} \arrow[ur] \arrow[dr] & & \mathrm{PS}_{\chi_3,\chi_2,\chi_1} \\ & \mathrm{PS}_{\chi_1,\chi_3,\chi_2} \arrow[ur] & & \mathrm{PS}_{\chi_3,\chi_1,\chi_2} \arrow[ur] & \end{tikzcd} \tag{*1}$$

with θ being the product of simple roots, the representations $\mathrm{PS}_{\chi'}$ defined as $\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)} \chi'(\varepsilon^{-1} \circ \theta)$. If ρ has Loewey length 3, we have $P_\rho = \widetilde{P}_\rho = B$ and the module diagram above is the initial on, i.e. all extensions are nonsplit.

Moreover, any representation Π compatible with B in the sense of [Bre+21, Def. 2.4.1.5] must have an initial labelled module diagram as above for some triple (χ_1, χ_2, χ_3).

Definition 4.8. For a representation Π compatible with B, we note Π^{ord} (resp. Π_{SS}, resp. Π_{ord}) the suquotients $\mathrm{soc}^2_{\mathrm{GL}_3(\mathbb{Q}_p)}\Pi$ (resp. $\mathrm{soc}^{3,\mathrm{GL}_3(\mathbb{Q}_p)}\Pi$, resp. $\Pi/\mathrm{soc}^3_{\mathrm{GL}_3(\mathbb{Q}_p)}\Pi$).

Remark 4.9. In [BH15], Beuil and Herzig constructed a representation $\Pi(\rho)^{\mathrm{ord}}$ that is conjectured to be the maximal subrepresentation of some $\Pi(\rho)$ constructed by global methods, with only principal serie Jordan-Hölder components. It coincides with Π^{ord} in our setting for any Π compatible with B and triple (χ_1, χ_2, χ_3) being the ordered Jordan-Hölder components of ρ.

We first explain why Zábrádi's functor applied to Π^{ord} recovers $\overline{L}^{\boxtimes}(\rho)^{\mathrm{ord}}$.

Definition 4.10. For a smooth character $\chi : T(\mathbb{Q}_p) \to \overline{\mathbb{F}_p}^{\times}$, we view it as a character of the product $(\mathrm{W}^{\mathrm{ab}}_{\mathbb{Q}_p} \times \mathrm{W}^{\mathrm{ab}}_{\mathbb{Q}_p} \times \mathbb{Q}_p^{\times})$ via

$$(\mathrm{W}^{\mathrm{ab}}_{\mathbb{Q}_p} \times \mathrm{W}^{\mathrm{ab}}_{\mathbb{Q}_p} \times \mathbb{Q}_p^{\times}) \xrightarrow{\sim} T(\mathbb{Q}_p), \ \ (g,h,x) \mapsto \mathrm{diag}(\mathrm{Art}^{-1}(gh)x,\, \mathrm{Art}^{-1}(h)x, x).$$

We restrict to $(\mathrm{W}_{\mathbb{Q}_p}^{\mathrm{ab}} \times \mathrm{W}_{\mathbb{Q}_p}^{\mathrm{ab}})$, extend to the profinite completion and inflate, obtaining a character χ_Δ of $(\mathcal{G}_{\mathbb{Q}_p} \times \mathcal{G}_{\mathbb{Q}_p})$.

Lemma 4.11. *Let $\chi : T(\mathbb{Q}_p) \to \overline{\mathbb{F}_p}^\times$ be a smooth character, seen by inflation as a character of $B^-(\mathbb{Q}_p)$. Then,*

$$\mathbf{V}_\Delta \left(\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)} \chi\right) = \chi_\Delta \otimes \delta_\Delta^{\boxtimes}.$$

Proof. For $G = T$, the set Δ_G is empty, the monoid T_+ of Zábrádi is $T(\mathbb{Q}_p)$ and the functor $\mathbf{D}_\emptyset^\vee$ sends a character on its dual seen as zero variable T_+-module. The result follows from [Záb18a, Th. 3.5] and our expression of $\mathbb{V}_\Delta^*$. $\square$

Proposition 4.12. *The functor $\mathbf{V}_\Delta$ establishes an isomorphism between*

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)} \left(\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)} s_\alpha(\chi)(\varepsilon^{-1} \circ \theta) \oplus \mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)} s_\beta(\chi)(\varepsilon^{-1} \circ \theta), \mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)} \chi(\varepsilon^{-1} \circ \theta)\right)$$

and

$$\mathrm{Ext}^1_{(\mathcal{G}_{\mathbb{Q}_p} \times \mathcal{G}_{\mathbb{Q}_p})} \big((\chi_2 \boxtimes (\chi_1\chi_2)) \oplus (\chi_1 \boxtimes (\chi_1\chi_3)), (\chi_1 \boxtimes (\chi_1\chi_2))\big).$$

The functor $\mathbf{V}_{\Delta,\xi}$ establishes a bijection between the same $\mathrm{GL}_3(\mathbb{Q}_p)$-Ext-group and

$$\mathrm{Ext}^1_{\mathcal{G}_{\mathbb{Q}_p}}(\chi_1\chi_2^2 \oplus \chi_1^2\chi_3, \chi_1^2\chi_2).$$

The same kind of result holds for

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)} \left(\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)} \chi^s(\varepsilon^{-1} \circ \theta), \mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)} s_\beta(\chi^s)(\varepsilon^{-1} \circ \theta) \oplus \mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)} s_\alpha(\chi^s)(\varepsilon^{-1} \circ \theta)\right)$$

with χ^s being the conjugate by the element of maximal length in the Weyl group of GL_3.

Proof. The previous lemma implies that

$$\mathbf{V}_\Delta(\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)} \chi(\varepsilon^{-1} \circ \theta)) = (\chi(\varepsilon^{-1} \circ \theta))_\Delta \otimes \delta_\Delta^{\boxtimes} = \chi_1 \boxtimes (\chi_1\chi_2)$$

and similarily for the others. Moreover, the genericity hypothesis and [Oll06, Th. 4] prove that all these inductions lie in SP^0. With this remark, it is a reformulation of [JZ25, Cor. 3.3.4].

Let's give the main ideas. First, the map is injective thanks to the faithfulness in Theorem 2.10. For essential surjectivity, we can do it independantly for the two extensions. Pick the first one. Thanks to [JZ25, Prop. 3.3.2], this extension must be trivial restricted to $\{\mathrm{Id}\} \times \mathcal{G}_{\mathbb{Q}_p}$. Genericity tell that the extension group is of dimension 1. Therefore [Hau16, Th. 1.1 ii)] produces a representation induced from the $\mathbb{Q}_p$-points of the parabolic P_α^- with Levi $\mathrm{GL}_2 \times \mathbb{G}_m$ which has the correct image. $\square$

Remark 4.13. It is an improvement, using Zábrádi's functor, of [Bre15, Th. 1.1] for $G = \mathrm{GL}_3$.

5 Representations results and finer failure on our toy example

We keep our representation ρ of dimension 3 and Loewy length 3.

Definition 5.1. We say that Π is weakly compatible with ρ if it is compatible with $\widetilde{P}_\rho$ and[13] if $\mathbf{V}_\xi(\Pi) \simeq \overline{L}^{\otimes}(\rho)$.

Lemma 5.2. *Let G be a profinite group and k an algebraically closed field. Let V and W be irreducible objects in $\mathrm{Rep}_k\, G$ and χ a 1-dimensional character. Then $\chi^{\oplus 2}$ can't be a subrepresentation of $V \otimes W$.*

[13] This is condition (i) in [Bre+21, Def. 2.4.2.7] but only for $\Pi' = \Pi$.

Proof. As everything is finite dimensional, we have

$$\begin{aligned}\mathrm{Hom}_{k[G]}(\chi, V\otimes W) &\simeq \mathrm{Hom}_{k[G]}\left((V\otimes W)^\vee, \chi^{-1}\right)\\ &\simeq \mathrm{Hom}_{k[G]}\left(V^\vee \otimes W^\vee, \chi^{-1}\right)\\ &\simeq \mathrm{Hom}_{k[G]}\left(V^\vee, W\otimes\chi^{-1}\right)\end{aligned}$$

As V is irreducible, so is $V^\vee$. As W is irreducible, so is $W\otimes\chi^{-1}$. Hence, Schur's lemma says that this $\mathrm{Hom}_{k[G]}$ is at most of 1-dimensional. It concludes. □

Lemma 5.3. *Let V belong to $\mathrm{Rep}_{\mathbb{F}}\mathcal{G}_{\mathbb{Q}_p}$ such that $\mathcal{I}_{\mathbb{Q}_p}^{\mathrm{wild}}$ acts trivially. Any subrepresentation of V that is an extension of two distinct characters is semisimple.*

Proof. Fix such subrepresentation W. Since any character is trivial when restricted to the pro-p-group $\mathcal{I}_{\mathbb{Q}_p}^{\mathrm{wild}}$, one can twist and suppose that W is an extension of 1 by χ, i.e. lives in $H^1(\mathcal{G}_{\mathbb{Q}_p}/\mathcal{I}_{\mathbb{Q}_p}^{\mathrm{wild}}, \chi)$. We reduce to proving that this group is zero when $\chi \not\equiv 1$. We use once again that

$$\mathcal{G}_{\mathbb{Q}_p}/\mathcal{I}_{\mathbb{Q}_p}^{\mathrm{wild}} \simeq \left(\varprojlim_{r\geq 1,\,\mathrm{norm\ map}} \mathbb{F}_{p^r}^\times\right) \rtimes \varphi^{\widehat{\mathbb{Z}}}$$

where the limit is identified with $\mathcal{I}_{\mathbb{Q}_p}/\mathcal{I}_{\mathbb{Q}_p}^{\mathrm{wild}}$.

The inflation-restriction sequence for $\mathcal{K} := (\mathcal{I}_{\mathbb{Q}_p}/\mathcal{I}_{\mathbb{Q}_p}^{\mathrm{wild}}) \rtimes \varphi^{\prod_{l\neq p}\mathbb{Z}_l}$ gives

$$0 \to H^1(\varphi^{\mathbb{Z}_p}, \chi^{\mathcal{K}}) \to H^1(\mathcal{G}_{\mathbb{Q}_p}/\mathcal{I}_{\mathbb{Q}_p}^{\mathrm{wild}}, \chi) \to H^1(\mathcal{K}, \chi_{|\mathcal{K}})$$

As $\mathcal{K}$ is pro-(prime to p), the last term vanishes. As $\mathbb{Z}_p$ is pro-p, we have $\chi_{|\varphi^{\mathbb{Z}_p}} \equiv 1$, our hypothesis $\chi\not\equiv 1$ implies $\chi_{|\mathcal{K}}\not\equiv 1$. We deduce that $\chi^{\mathcal{K}} = \{0\}$ and the first term also vanishes. □

Corollary 5.4. *Let V be an irreducible object in $\mathrm{Rep}_{\overline{\mathbb{F}_p}}(\mathcal{G}_{\mathbb{Q}_p}\times\mathcal{G}_{\mathbb{Q}_p})$. Any $W\subset \mathrm{Res}_{\mathcal{G}_{\mathbb{Q}_p}}^{(\mathcal{G}_{\mathbb{Q}_p}\times\mathcal{G}_{\mathbb{Q}_p})} V$ which is an extension of two distincts characters is semisimple.*

Proof. Such irreducible V must be written $V_1\boxtimes V_2$ where V_i are irreducible objects in $\mathrm{Rep}_{\overline{\mathbb{F}_p}}\mathcal{G}_{\mathbb{Q}_p}$. It is a general fact that any normal pro-p-subgroup of a profinite group acts trivially on any irreducible representation over a (inductive limit) of finite characteristic p fields. Both V_1 and V_2 factorise by $\mathcal{I}_{\mathbb{Q}_p}^{\mathrm{wild}}$. Thus, Lemma 5.3 applied to $\mathrm{Res}_{\mathcal{G}_{\mathbb{Q}_p}}^{(\mathcal{G}_{\mathbb{Q}_p}\times\mathcal{G}_{\mathbb{Q}_p})} V$ concludes. □

We're ready for our second result.

Proposition 5.5. *Let ρ belong to $\mathrm{Rep}_{\overline{\mathbb{F}_p}}\mathcal{G}_{\mathbb{Q}_p}$, of length 3 and Loewy length 3, generic and good conjugate. Let Π be weakly compatible with ρ.*

1. *We have* $\dim \mathbf{V}_\Delta(\Pi)\leq 5$.
2. *We have* $\overline{L}^{\boxtimes}(\rho)^{\mathrm{ord}} = \mathbf{V}_\Delta(\Pi^{\mathrm{ord}}) \hookrightarrow \mathbf{V}_\Delta(\Pi)$.
3. *If $p\neq 2$, we have* $\dim\mathbf{V}_\Delta(\Pi)\leq 4$, *iff* $\mathbf{V}_\Delta(\Pi)\to\mathbf{V}_\Delta(\Pi_{\mathrm{ord}})$ *is nonzero.*

The first point implies that $\mathbf{V}_{\Delta,\xi}(\Pi)\not\simeq \overline{L}^{\otimes}(\rho)$.

Proof. We have an exact sequence

$$0\to\mathbf{V}_\Delta(\Pi^{\mathrm{ord}})\to\mathbf{V}_\Delta(\Pi)\to\mathbf{V}_\Delta(\Pi/\Pi^{\mathrm{ord}}).$$

In particular, thanks to weak compatibility and Corollary 2.15, we have $\mathbf{V}_{\Delta,\xi}(\Pi^{\mathrm{ord}})\hookrightarrow \overline{L}^{\otimes}(\rho)$. Using the second bijection of Proposition 4.12, we get a labelle module diagram for $\mathbf{V}_{\Delta,\xi}(\Pi^{\mathrm{ord}})$ looking like

$$\begin{tikzcd}
 & \chi_1^2\chi_3 \\
\chi_1^2\chi_2 \arrow[ur] \arrow[dr] & \\
 & \chi_1\chi_2^2
\end{tikzcd}$$

In particular, genericty implies that it does not have $(\det\rho)$ as Jordan-Hölder component. Thanks to Corollary 4.6, any nonzero subrepresentation of $\overline{L}^{\otimes}/\overline{L}^{\otimes}(\rho)^{\mathrm{ord}}$ has $\det\rho$ in its socle. We deduce that the image of $\mathbf{V}_{\Delta,\xi}(\Pi^{\mathrm{ord}})$ sits in $\overline{L}^{\otimes}(\rho)^{\mathrm{ord}}$. As they have same length, it is exactly $\overline{L}^{\otimes}(\rho)^{\mathrm{ord}}$.

Consider $W := \mathbf{V}_{\Delta,\xi}(\Pi)/\mathbf{V}_{\Delta,\xi}(\Pi^{\mathrm{ord}})$. Weak compatility and the above isomorphism show that it is a subrepresentation of $\overline{L}^{\otimes}(\rho)/\overline{L}^{\otimes}(\rho)^{\mathrm{ord}}$. We deduce that $\mathrm{soc}_{\mathcal{G}_{\mathbb{Q}_p}}(W) \to \mathrm{soc}_{\mathcal{G}_{\mathbb{Q}_p}}(\overline{L}^{\otimes}(\rho)/\overline{L}^{\otimes}(\rho)^{\mathrm{ord}})$ is an injection, hence $\mathrm{soc}_{\mathcal{G}_{\mathbb{Q}_p}}(W)$ is a direct sum of $(\det\rho)$. Left exactness of $\mathbf{V}_{\Delta,\xi}$ also implies that $\mathrm{soc}_{\mathcal{G}_{\mathbb{Q}_p}}(W)$ is a subrepresentation of $\mathbf{V}_{\Delta,\xi}(\Pi/\Pi^{\mathrm{ord}})$. This last representation sits in an exact sequence

$$0 \to \mathbf{V}_{\Delta,\xi}(\Pi_{\mathrm{SS}}) \to \mathbf{V}_{\Delta,\xi}(\Pi/\Pi^{\mathrm{ord}}) \to \mathbf{V}_{\Delta,\xi}(\Pi_{\mathrm{ord}}).$$

Using the last part of Proposition 4.12 to compute $\mathbf{V}_{\Delta,\xi}(\Pi_{\mathrm{ord}})$, $\det\rho$ is not one of its Jordan-Hölder factors, yielding that $\mathrm{soc}_{\mathcal{G}_{\mathbb{Q}_p}} W$ is a subrepresentation of $\mathbf{V}_{\Delta,\xi}(\Pi_{\mathrm{SS}})$. Theorem 2.11 says that $\mathbf{V}_{\Delta,\xi}(\Pi_{\mathrm{SS}})$ is the restriction of an irreducible representation of $(\mathcal{G}_{\mathbb{Q}_p}\times\mathcal{G}_{\mathbb{Q}_p})$, hence has multiplicity one socle by Lemma 5.2. It proves that $\mathrm{soc}_{\mathcal{G}_{\mathbb{Q}_p}} W$ is one dimensional. As W is a subrepresentation of $\overline{L}^{\otimes}(\rho)/\overline{L}^{\otimes}(\rho)^{\mathrm{ord}}$, the description in Corollary 4.6 proves that it is at most 2-dimensional which finishes to prove the first claim.

It also proves that if $\dim W = 2$, the representation W is an extension of two distincts (thanks to genericity) characters. Using Corollary 5.4, it can't be a subrepresentation of $\mathbf{V}_{\Delta,\xi}(\Pi_{\mathrm{SS}})$, which proves the third claim.

Finally, use simultaneously the two first isomorphisms of Proposition 4.12 to show that $\mathbf{V}_{\Delta,\xi}(\Pi^{\mathrm{ord}}) = \overline{L}^{\otimes}(\rho)^{\mathrm{ord}}$ implies $\mathbf{V}_{\Delta}(\Pi^{\mathrm{ord}}) = \overline{L}^{\boxtimes}(\rho)^{\mathrm{ord}}$. □

Remark 5.6. We used nothing more that weak compatibility. With (stronger) compatibility in the sense of [Bre+21], we would obtain directly that $\mathbf{V}_{\Delta,\xi}(\Pi_{\mathrm{SS}})$ is 1-dimensional. I don't see how it may improve the dimension bound.

Remark 5.7. Be careful that Lemma 5.2 breaks for $\#\Delta \geq 3$. In this case, one could strengthen the weak compatibility by replacing $\mathbf{V}_{\xi}(\Pi) \simeq \overline{L}^{\otimes}(\rho)$ by $\mathbf{V}_{\Delta}(\Pi) \subset \overline{L}^{\boxtimes}(\rho)$. This also give precise bounds of the dimension. For ρ of dimension, length and Loewy length 4, it gives $\dim\mathbf{V}_{\Delta}(\Pi) \leq 9$.

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