Computations of (φ, Γ)-modules for some non semisimple representations modulo p^2

Nataniel Marquis

October 6, 2025

This text aims to compute Fontaine's functor for specific representations. We consider several variations around the same example of a $\mathbb{Z}_p$-representation V whose underlying $\mathbb{Z}_p$-module is isomorphic to $(\mathbb{Z}/p^2\mathbb{Z} \oplus \mathbb{F}_p)$ but for which $V[p]$ isn't semi-simple as Galois representation.

In this text, the prime p is odd.

1 First univariable case

Consider $E = \mathbb{F}_p((X))$. Following the notations of [Fon07, §1.2], we fix $\mathcal{O}_{\mathcal{E}}$ a p-adically complete and separated $\mathbb{Z}_p$-algebra, discrete valuation ring with uniformiser p and residue field E, equipped with a lift of Frobenius. We also denote by $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$ its strict henselisation.

Fix a non trivial character

$$\chi : \mathcal{G}_E \to \mathbb{F}_p$$

and define $F|E$ to be the Galois extension of degree p corresponding to $\mathrm{Ker}(\chi)$. Consider V_χ the $\mathbb{Z}_p$-representation of $\mathcal{G}_E$ whose underlying $\mathbb{Z}_p$-module is

$$\left(\mathbb{Z}/p^2\mathbb{Z}\right) e_1 \oplus \mathbb{F}_p e_2$$

and whose action is given by

$$\begin{aligned} \sigma \cdot e_1 &= e_1 \\ \sigma \cdot e_2 &= p\chi(\sigma) e_1 + e_2 \end{aligned}$$

One feature of this representation, is that $V_\chi[p]$ has (pe_1, e_2) for basis and that the Galois action in this basis expresses as

$$\begin{pmatrix} 1 & \chi(\sigma) \\ 0 & 1 \end{pmatrix}.$$

Thus $V_\chi[p]$ isn't semi-simple.

We compute the $\varphi^{\mathbb{N}}$-module over $\mathcal{O}_{\mathcal{E}}$

$$\begin{aligned} \mathbb{D}(V_\chi) &:= \left(\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathcal{O}_{\mathcal{E}}} V_\chi\right)^{\mathcal{G}_E} \\ &\cong \left\{ (x, y) \in \left(\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}/p^2\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}\right) \times E^{\mathrm{sep}} \;\middle|\; \forall \sigma \in \mathcal{G}_E, \begin{array}{c} \sigma(x) + p\chi(\sigma)\sigma(y) = x \\ \sigma(y) = y \end{array} \right\} \end{aligned}$$

These equalities imply that x is invariant by $\mathcal{G}_F$ and y by $\mathcal{G}_E$. Thus, they respectively land in ${}^{\mathcal{O}_{\mathcal{F}}}/_{p^2\mathcal{O}_{\mathcal{F}}}$ and E. The group $\mathrm{Gal}\,(F|E)$ is cyclic ; call $\Sigma := \chi^{-1}(1)_{|F}$ which is a generator . We obtain that

$$\mathbb{D}(V_\chi) \cong \left\{ (x,y) \in \left({}^{\mathcal{O}_{\mathcal{F}}}/_{p^2\mathcal{O}_{\mathcal{F}}}\right) \times E \;\middle|\; \Sigma(x) + py = x \right\}.$$

We use property A.1 on $F|E$ to produce an element $Y_\diamond \in F$ such that $\Sigma(Y_\diamond) + 1 = Y_\diamond$. We then compute that

$$\mathbb{D}(V_\chi) = \left({}^{\mathcal{O}_{\mathcal{E}}}/_{p^2\mathcal{O}_{\mathcal{E}}}\right) d_1 \oplus E d_2$$

where $d_1 = e_1$ and $d_2 = pY_\diamond e_1 + e_2$.

Nonetheless, computing the Frobenius in this case requires an understanding of F in terms of Artin-Schreier theory (see [Neu99, Chapter IV, §3] for $\wp = \varphi - \mathrm{Id}$). Precisely, we use

Theorem 1.1 (Artin-Schreier). *Let E be a field of characteristic p and $\wp = \varphi - \mathrm{Id}$. The map*

$$E/_{\wp(E)} \to \mathrm{Hom}_{\mathrm{TopGp}}\left(\mathcal{G}_E, \mathbb{F}_p\right), \;\; x + \wp(E) \mapsto [\sigma \mapsto \sigma(y) - y],$$

for any y such that $\wp(y) = x$ is well defined. It is an linear isomorphism.

Using that $Y_\diamond \in F$ and $\Sigma(Y_\diamond) - Y_\diamond = -1$, we know that

$$\chi = [\sigma \mapsto Y_\diamond - \sigma(Y_\diamond)].$$

The Artin-Shreier theory says that $X_\diamond := Y_\diamond^p - Y_\diamond \in E$, that χ is associated $-X_\diamond + \wp(E)$ and F is the decomposition field of $T^p - T + X_\diamond$. With these notations, we obtain

$$\begin{aligned} \varphi(d_1) &= d_1 \\ \varphi(d_2) &= pX_\diamond d_1 + d_2 \end{aligned}$$

Note that the character χ (i.e. the extension $F|E$ with generator of its Galois group) entirely determines $X_\diamond + \wp(E)$. Changing the representative only modify the given base $\mathbb{D}(V_\chi)$.

2 Second univariable case

In the previous setup, consider W_χ the $\mathbb{Z}_p$-representation of $\mathcal{G}_E$ whose underlying $\mathbb{Z}_p$-module is

$$\left({}^{\mathbb{Z}}/_{p^2\mathbb{Z}}\right) e_1 \oplus \mathbb{F}_p e_2$$

and whose action is given by

$$\begin{aligned} \sigma \cdot e_1 &= e_1 + \chi(\sigma) e_2 \\ \sigma \cdot e_2 &= p\chi(\sigma) e_1 + e_2 \end{aligned}$$

Like $V_\chi[p]$, the representation $W_\chi[p]$ isn't semi-simple. It is even funnier : where V_χ had a $\mathcal{G}_E$-stable submodule isomorphic to ${}^{\mathbb{Z}}/_{p^2\mathbb{Z}}$, the representation W_χ has none. Indeed,

$$\forall \lambda \in \mathbb{F}_p, \;\; \sigma(e_1 + \lambda e_2) = (1 + p\lambda\chi(\sigma))\,(e_1 + (\lambda + \chi(\sigma))e_2).$$

We compute the $\varphi^{\mathbb{N}}$-module over $\mathcal{O}_{\mathcal{E}}$

$$\begin{aligned} \mathbb{D}(W_\chi) &:= \left(\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathcal{O}_{\mathcal{E}}} W_\chi\right)^{\mathcal{G}_E} \\ &\cong \left\{ (x,y) \in \left({}^{\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}}/_{p^2\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}}\right) \times E^{\mathrm{sep}} \;\middle|\; \forall \sigma \in \mathcal{G}_E, \; \begin{array}{c} \sigma(x) + p\chi(\sigma)\sigma(y) = x \\ \sigma(y) + \chi(\sigma)\sigma(x \bmod p) = y \end{array} \right\} \end{aligned}$$

These equalities for σ in the absolute Galois group of F give that both x and y are invariant by $\mathcal{G}_F$. Thus, they respectively land in ${}^{\mathcal{O}_\mathcal{F}}/{}_{p^2\mathcal{O}_\mathcal{F}}$ and F. The group $\mathrm{Gal}\,(F|E)$ is cyclic ; call $\Sigma := \chi^{-1}(1)_{|F}$ which is a generator . We obtain that

$$\mathbb{D}(W_\chi) \cong \left\{ (x,y) \in \left({}^{\mathcal{O}_\mathcal{F}}/{}_{p^2\mathcal{O}_\mathcal{F}}\right) \times F \;\middle|\; \Sigma(x) + p\Sigma(y) = x \text{ and } \Sigma(y) + \Sigma(x \bmod p) = y \right\}.$$

We use property A.1 on $F|E$ to produce a sequence $(1, Y_\diamond, Y_\blacklozenge)$ in F such that $\Sigma(Y_\diamond) + 1 = Y_\diamond$ and $\Sigma(Y_\blacklozenge) + \Sigma(Y_\diamond) = Y_\blacklozenge$. Then, we compute that

$$\mathbb{D}(V_\chi) = \left({}^{\mathcal{O}_\mathcal{E}}/{}_{p^2\mathcal{O}_\mathcal{E}}\right) d_1 \oplus E d_2$$

where $d_1 = (1 + pY_\blacklozenge)e_1 + Y_\diamond e_2$ and $d_2 = pY_\diamond e_1 + e_2$.

Again, the Artin-Shreier theory says that $X_\diamond := Y_\diamond^p - Y_\diamond \in E$, that χ is associated $-X_\diamond + \wp(E)$ and F is the decomposition field of $T^p - T + X_\diamond$. We also compute

$$\begin{aligned}
\Sigma(Y_\blacklozenge^p - Y_\blacklozenge) - (Y_\blacklozenge^p - Y_\blacklozenge) &= \Sigma(Y_\blacklozenge)^p - \Sigma(Y_\blacklozenge) - (Y_\blacklozenge^p - Y_\blacklozenge) \\
&= (Y_\blacklozenge - Y_\diamond + 1)^p - (Y_\blacklozenge - Y_\diamond + 1) - (Y_\blacklozenge^p - Y_\blacklozenge) \\
&= -(Y_\diamond^p - Y_\diamond) \\
&= -X_\diamond
\end{aligned}$$

Our analysis of $(\Sigma - \mathrm{Id})$ implies that

$$\exists X_\blacklozenge \in E, \;\; Y_\blacklozenge^p - Y_\blacklozenge = X_\diamond Y_\diamond + X_\blacklozenge.$$

With these notations, we obtain

$$\begin{aligned}
\varphi(d_1) &= (1 + pX_\blacklozenge)d_1 + X_\diamond d_2 \\
\varphi(d_2) &= pX_\diamond d_1 + d_2
\end{aligned}$$

Note that the character χ (i.e. the extension $F|E$ with generator of its Galois group) entierely determines $X_\diamond + \wp(E)$ and $X_\blacklozenge + E^p X_\diamond + \wp(E)$. Any compatible choice of representatives only change the given base $\mathbb{D}(W_\chi)$.

3 First multivariable case

We place in the setting of [CKZ21], once again for $E = \mathbb{F}_p((X))$. This article fixes a finite set Δ and construct a multivariable version of Fontaine's equivalence. All indexes α on objects stand for a labelled copy of the corresponding object in the univariable case. The three authors construct rings

$$E_\Delta := \mathbb{F}_p[\![X_\alpha \,|\, \alpha \in \Delta]\!][X_\Delta^- 1]$$

where $X_\Delta = \prod X_\alpha$,

$$\mathcal{O}_{\mathcal{E}_\Delta} := \left(\mathbb{Z}_p[\![X_\alpha \,|\, \alpha \in \Delta]\!][X_\Delta^- 1]\right)^{\wedge p}$$

and a ring $\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{\mathrm{ur}}}}$.

Each E_α (resp. each $\mathcal{O}_{\mathcal{E}_\alpha}$, resp. each $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}_\alpha}}}$) embeds into E_Δ (resp. $\mathcal{O}_{\mathcal{E}_\Delta}$, resp. $\mathcal{O}_{\widehat{\mathcal{E}_\Delta^{\mathrm{ur}}}}$). The three rings are equiped with a linear action of the monoid $\Phi_{\Delta,p} := \prod \varphi_{\alpha,p}^{\mathbb{N}}$ satisfying

$$\varphi_{\alpha,p}(X_\alpha) = (1 + X_\alpha)^p - 1 \;\text{ and }\; \forall \beta \neq \alpha, \;\; \varphi_{\beta,p}(X_\alpha) = X_\alpha.$$

The ring $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$ is also equiped with an action of $\mathcal{G}_{E,\Delta} := \prod \mathcal{G}_{E_\alpha}$, acting via the quotient $\mathcal{G}_{E_\alpha}$ on $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}_\alpha}}$ as predicted. The two actions commutes and

$$\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}_\Delta}}^{\mathcal{G}_{E,\Delta}} = \mathcal{O}_{\mathcal{E}_\Delta}, \;\; \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}_\Delta}}^{\Phi_{\Delta,p}} = \mathbb{Z}_p.$$

Now, we consider the functor

$$\mathbb{D}_\Delta \,:\, \mathrm{Rep}_{\mathbb{Z}_p}\mathcal{G}_{E,\Delta} \to \mathrm{Mod}(\Phi_{\Delta,p}, \mathcal{O}_{\mathcal{E}_\Delta}), \;\; V \mapsto \left(\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}_\Delta}} \otimes_{\mathbb{Z}_p} V\right)^{\mathcal{G}_{E,\Delta}}$$

from finite type continuous $\mathbb{Z}_p$-representations of $\mathcal{G}_{E,\Delta}$ to $\mathcal{O}_{\mathcal{E}}$-modules with a semilinear action of $\Phi_{\Delta,p}$. Our goal is to compute this functor in one case.

Take $\Delta = \{\alpha, \beta\}$. Choose a pair of continuous characters

$$\underline{\chi} = (\chi_\alpha \,:\, \mathcal{G}_{E_\alpha} \to \mathbb{F}_p, \, \chi_\beta \,:\, \mathcal{G}_{E_\beta} \to \mathbb{F}_p).$$

Call F_α and F'_β their kernels, which are extensions of degree p of E_α and E_β. Consider $V_{\underline{\chi}}$ the $\mathbb{Z}_p$-representation of $\mathcal{G}_{E,\Delta}$ whose underlying $\mathbb{Z}_p$-module is

$$\left(\mathbb{Z}/p^2\mathbb{Z}\right) e_1 \oplus \mathbb{F}_p e_2$$

and whose action is given by

$$\sigma_\alpha \cdot e_1 = (1 + p\chi_\alpha(\sigma_\alpha))e_1 \quad\;\; \sigma_\beta \cdot e_1 = e_1 + \chi_\beta(\sigma_\beta)e_2$$
$$(\sigma_\alpha, \sigma_\beta) \cdot e_2 = e_2$$

Like $V_\chi[p]$, the representation $V_{\underline{\chi}}[p]$ isn't semi-simple and $V_{\underline{\chi}}$ has no $\mathcal{G}_{\mathbb{Q}_p}$-stable submodule isomorphic to ${}^{Z}/{}_{p^2\mathbb{Z}}$. Indeed,

$$\forall \lambda \in \mathbb{F}_p, \;\; \sigma_\beta(e_1 + \lambda e_2) = e_1 + (\lambda + \chi_\beta(\sigma_\beta))e_2.$$

We compute the $\Phi_{\Delta,p}$-module over $\mathcal{O}_{\mathcal{E}_\Delta}$

$$\begin{aligned}\mathbb{D}_\Delta(V_{\underline{\chi}}) &:= \left(\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}_\Delta}} \otimes_{\mathcal{O}_{\mathcal{E}_\Delta}} V_{\underline{\chi}}\right)^{\mathcal{G}_{E,\Delta}} \\ &\cong \left\{ (x,y) \in \left(\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}_\Delta}}/p^2\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}_\Delta}}\right) \times E_\Delta^{\mathrm{sep}} \;\middle|\; \forall \sigma \in \mathcal{G}_{E,\Delta}, \; \begin{array}{r} (1 + p\chi_\alpha(\sigma_\alpha))\sigma_\alpha(x) = x \\ \sigma_\alpha(y) = y \\ \sigma_\beta(x) = x \\ \sigma_\beta(y) + \chi_\beta(\sigma_\beta)(\sigma_\beta(x) \bmod p) = y \end{array} \right\}\end{aligned}$$

Considering $\sigma_\alpha \in \mathcal{G}_{E_\alpha}$ or $\mathcal{G}_{F_\alpha}$ and $\sigma_\beta \in \mathcal{G}_{E_\beta}$ or $\mathcal{G}_{F'_\beta}$ and using [Záb18, Lemma 3.2 and Proposition 3.3] we found that

$$x \in \left(\mathcal{O}_{\mathcal{F}_\alpha} \otimes_{\mathcal{O}_{\mathcal{E}_\alpha}} \mathcal{O}_{\mathcal{E}_\Delta}\right)/p^2 \;\text{ and }\; y \in E_\Delta \otimes_{E_\beta} F'_\beta.$$

As in the first univariable case, we call Σ_α the generator of $\mathrm{Gal}\,(F_\alpha|E_\alpha)$ pinned down by χ_α. Take $Y_{\diamond,\alpha} \in F_\alpha$ such that $\Sigma_\alpha(Y_{\diamond,\alpha}) + 1 = Y_{\diamond,\alpha}$ and $X_{\diamond,\alpha} := Y_{\diamond,\alpha}^p - Y_{\diamond,\alpha} \in E_\alpha$. Likewise, define Σ_β, $Y'_{\diamond,\beta} \in F'_\beta$ and $X_{\diamond,\beta} \in E_\beta$. We obtain

$$\mathbb{D}_\Delta(V_{\underline{\chi}}) = \left(\mathcal{O}_{\mathcal{E}_\Delta}/p^2\mathcal{O}_{\mathcal{E}_\Delta}\right) d_1 \oplus E_\Delta d_2$$

where $d_1 = (1 + pY_{\diamond,\alpha})e_1 + Y'_{\diamond,\beta}e_2$ and $d_2 = e_2$. Remark that pe_1 belongs to this module and is equal to pd_1. We also have

$$\begin{aligned} \varphi_{\alpha,p}(d_1) &= (1 + pX_{\diamond,\alpha})d_1 \\ \varphi_{\beta,p}(d_1) &= d_1 + X_{\diamond,\beta}d_2 \\ \varphi_{\alpha,p}(d_2) = \varphi_{\beta,p}(d_2) &= d_2 \end{aligned}$$

A One useful proposition

We prove the proposition about cyclic p-prime extension of characteristic p fields.

Proposition A.1. *Let p be a prime and Let E be a field of characteristic p and $F|E$ be a cyclic Galois extension of order p^n for some $n \geq 1$. Let τ be a generator of $\mathrm{Gal}\,(F|E)$. If $k \leq p^n$ and $x_1 \in E$, there exists a sequence $(x_2, \dots, x_k) \in F^k$ such that*

$$\forall 1 \leq i < k,\ \tau(x_{i+1}) - x_{i+1} = x_i.$$

Proof. Considerer the map

$$\mathrm{T} \,:\, F \to F,\ x \mapsto \tau(x) - x.$$

It is an E-linear endomorphism of a p^n-dimensional E-vector space. Because the group $\mathrm{Gal}\,(F|E)$ is cyclic generated by τ, the kernel of T equals E (hence one dimensional). Moreover,

$$\mathrm{T}^{\circ p^n} = (X-1)^{p^n}(\tau) = \tau^{\circ p^n} - \mathrm{Id}_F = 0.$$

The endomorphism T is nilpotent of order precisely p^n thanks to the decreasing of the dimensional gaps in the iterated kernel sequence. Jordan decomposition concludes. □

References

[Neu99] Jürgen Neukirch. *Algebraic Number Theory*. Vol. 322. Grundlehren der mathematischen Wissenschaften. Springer Link, 1999.

[Fon07] Jean-Marc Fontaine. “Représentations p-adiques des corps locaux, $1^{\text{ère}}$ partie”. In: *The Grothendieck Festschrift*. Vol. II. Birkhäuser, Boston, MA, 2007, pp. 249–309.

[Záb18] Gergely Zábrádi. “Multivariable (φ, Γ)-modules and products of Galois groups”. In: *Math. Res. Lett.* 25(2) (2018), pp. 687–721.

[CKZ21] Annie Carter, Kiran Kedlaya, and Gergely Zábrádi. “Drinfeld’s lemma for perfectoid spaces and overconvergence of multivariate (φ, Γ)-modules”. In: *Documenta Math.* (2021), pp. 1329–1393.