Field isomorphisms of p-adic fields

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Let p be a prime. Using [Rib99, Chapter 3, X], we can obtain that any field isomorphism between p-adic fields (i.e. finite extensions of $\mathbb{Q}_p$) is continuous. This short note aims to give an alternative proof using only the structure of the description of the multiplicative group of these fields given in [Neu99, Chapter II, Proposition 5.7].

Lemma 0.1. *Let K be a p-adic field and q the cardinal of its residue field. Then*

$$u \in \mathcal{O}_K^\times \textit{ iff } u^{q-1} \textit{ has } n\textit{-th roots for all } \gcd(n,p)=1.$$

Proof. We have an isomorphism

$$K^\times \cong \pi^{\mathbb{Z}} \times \mu_{q-1} \times \mu_{p^\infty}(K) \times \mathbb{Z}_p^d$$

where μ_{q-1} are the $(q-1)$-th roots of unity and $\mu_{p^\infty}(K)$ is the subgroup of roots of order a power of p, π is a uniformizer and the pre-image of $\mathbb{Z}_p^d$ is included into $\mathcal{O}_K^\times$. Any element of $\mu_{p^\infty}(K) \times \mathbb{Z}_p^d$ has roots of any order prime to p. Any non-trivial element of $\pi^{\mathbb{Z}}$ doesn't have all roots of order prime to p. We conclude by noticing that $u^{q-1} \in \pi^{\mathbb{Z}} \times \mu_{p^\infty}(K) \times \mathbb{Z}_p^d$ with trivial component on $\pi^{\mathbb{Z}}$ iff u belonged to $\mathcal{O}_K^\times$. □

Lemma 0.2. *Let K be a p-adic field. Then*

$$u \textit{ is a uniformizer or an inverse of a uniformizer iff it is of infinite order and } u^{\mathbb{Z}} \subset K^\times \textit{ is split.}$$

Proof. We decompose again

$$K^\times \cong \pi^{\mathbb{Z}} \times \mu_\infty(K) \times \mathbb{Z}_p^d$$

and notice that $\mu_\infty(K)$ is the torsion part of $K^\times$. Write $u = (\pi^n, \zeta, v)$. If u is not of finite order then $n \neq 0$ or $v \neq 0$ thanks to.

If u is a uniformizer or an inverse of such, $n = \pm 1$. Thus, $\mathcal{O}_K^\times \subset K^\times$ gives the desired splitting.

If $|n| \geq 2$ the quotient $K^\times / u^{\mathbb{Z}}$ has torsion isomorphic to $\mathbb{Z}/n\mathbb{Z} \times \mu_\infty(K)$ hence the exact sequence cannot be split.

If $n = 0$, then $v \neq 0$ and upon chosing a correct basis of (e_i) of $\mathbb{Z}_p^d$ we can suppose that $v = p^k e_1$ for $k \geq 0$. Then, for $x \in \mathbb{Z}_p \backslash \mathbb{Z}$, the image of $p^k v e_1$ is infinitely divisible in $K^\times / u^{\mathbb{Z}}$. Hence the exact sequence is not split. □

Proposition 0.3. *Any field isomorphism of p-adic fields is continuous.*

Proof. Let f be a field isomorphism of a p-adic field K. By the first lemma $f(\mathcal{O}_K^\times) = \mathcal{O}_K^\times$. Let π be a uniformizer. By the second lemma $f(\pi)$ or $f(\pi)^{-1}$ is a uniformizer. Writing $p = \pi^\nu u$ with $\nu \geq 0$ and $u \in \mathcal{O}_K^\times$ we compute

$$1 \geq |p|_K = |f(p)|_K = |f(\pi)^\nu f(u)|_K = |f(\pi)|_K^\nu.$$

Hence, $f(\pi)$ is a uniformizer. This finishes to prove that f preserves the norm. □

References

[Neu99] Jürgen Neukirch. *Algebraic Number Theory*. Vol. 322. Grundlehren der mathematischen Wissenschaften. Springer Link, 1999.

[Rib99] Paul Ribenboim. *The Theory of Classical Valuations*. Springer Monographs in Mathematics. Springer, 1999.